

# CHARACTERISTIC CLASSES OF QUANTUM PRINCIPAL BUNDLES

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ABSTRACT. A noncommutative-geometric generalization of classical Weil theory of characteristic classes is presented, in the conceptual framework of quantum principal bundles. A particular care is given to the case when the bundle does not admit regular connections. A cohomological description of the domain of the Weil homomorphism is given. Relations between universal characteristic classes for the regular and the general case are analyzed. In analogy with classical geometry, a natural spectral sequence is introduced and investigated. The appropriate counterpart of the Chern character is constructed, for structures admitting regular connections. Illustrative examples and constructions are presented.

## CONTENTS

<b>1. Introduction</b>	<b>2</b>
<b>2. General Conceptual Framework</b>	<b>5</b>
<b>3. Quantum Characteristic Classes</b>	<b>11</b>
3.1. The Regular Case	
3.2. Some Variations	
3.3. The Spectral Sequence	
3.4. The Universal Construction	
<b>4. Quantum Chern Character</b>	<b>28</b>
4.1. The Canonical Trace	
4.2. Associated $K$ -rings & Chern Characters	
4.3. Explicit Expressions	
4.4. Chern Classes	
<b>5. Quantum Line Bundles</b>	<b>37</b>
5.1. Reconstruction Problematics	
5.2. Differential Structures and The Euler Class	
<b>6. Bundles Over Classical Manifolds</b>	<b>41</b>
<b>7. Framed Bundles</b>	<b>43</b>
<b>8. Concluding Examples &amp; Remarks</b>	<b>44</b>
<b>References</b>	<b>48</b>

## 1. INTRODUCTION

The theory of characteristic classes plays a fundamental role in understanding the internal structure of classical spaces. In classical differential geometry, the Weil approach [11] to characteristic classes is the most natural. Characteristic classes are labeled by invariant polynomials on the Lie algebra of the structure group. A connection with the purely topological approach is given by establishing the isomorphism between the algebra of invariant polynomials and the cohomology algebra of the classifying space of the structure Lie group.

The aim of this study is to incorporate classical Weil theory of characteristic classes into a general quantum framework, in accordance with basic principles of non-commutative differential geometry [1, 2]. Considerations of this study are logically based on a general theory of quantum principal bundles, developed in [5, 6].

Let us outline the contents of the paper. In the next section the main facts from the theory of quantum principal bundles are collected. This includes a brief exposition of differential calculus, and the formalism of connections, together with the description of main associated algebras. In particular, general constructions of operators of covariant derivative, horizontal projection and curvature are sketched. Finally, the structure of the algebra of horizontal forms is analyzed, from the point of view of the representation theory of the structure group.

General theory of characteristic classes is presented in Section 3. As first, a construction of the Weil homomorphism given in [6] is sketched. This works fine when the bundle admits regular and multiplicative connections. However it turns out that essentially the same construction can be performed in a more general context, when the algebra of horizontal forms and the appropriate space of “vertical” forms are mutually transversal. Next, we shall discuss a question of renormalizing the calculus on the bundle so that an arbitrarily given connection becomes regular, in terms of the renormalized calculus. Such a construction always works, however the resulting calculus may be too simple, if the initial connection is sufficiently irregular.

We shall then introduce and analyze a natural spectral sequence, in analogy with classical geometry. This spectral sequence is associated to the filtration of the calculus on the bundle, induced by the pull back of the right action map. The corresponding first two cohomology algebras will be analyzed in details.

The last and the main theme of Section 3 is a presentation of a universal approach to characteristic classes, which gives a cohomological interpretation of the domain of the Weil homomorphism, incorporating certain elements of classical classifying space interpretation of universal characteristic classes (as cohomology classes of the corresponding classifying space).

The construction works for arbitrary quantum principal bundles and connections. It also admits a natural “projection” to the level of bundles admitting regular connections. Interestingly, although the two levels of the theory (regular and general) are formulated in essentially the same way, the corresponding characteristic classes radically differ in sufficiently “irregular” situations.

For example, at the level of general bundles there exist generally nontrivial classes in *odd* dimensions, in contrast to the regular case where all classes are expressible in terms of the curvature map. Another important difference between two levels is in the existence of the Chern-Simons forms. At the general level, all characteristic

classes vanish, as cohomology classes of the bundle. In contrast to this, there exist regular characteristic classes, nontrivial as classes on the bundle. In particular, such classes have no analogs at the level of general differential structures. However, it turns out that under certain regularity assumptions on the calculus over the structure group, universal classes for the regular and the general case coincide. This essentially simplifies the work with bundles without regular connections.

The basic theme of Section 4 is the construction of the quantum Chern character. It is essential for the construction that the bundle admits regular connections. The first step is to construct a pairing between vector bundles associated to a given quantum principal bundle  $P$ , and differential forms from the graded centre  $Z(M)$  of the algebra  $\Omega(M)$  representing the calculus on the base manifold  $M$ . Associated vector bundles will be represented by intertwiner  $\Omega(M)$ -bimodules, between representations of the structure group  $G$  and the right action of  $G$  on horizontal forms (so that the elements of these bimodules are interpretable as vector-bundle valued forms on  $M$ ). The above mentioned pairing will be constructed with the help of a canonical “trace” on the corresponding vector bundle endomorphism algebras, and the covariant derivative of an arbitrary regular connection.

The next step in the construction is to pass to the corresponding cohomology algebra  $HZ(M)$ . This rules out the connection-dependence of the pairing. Finally, starting from intertwining bimodules (with natural operations of addition and tensor product over  $\Omega(M)$ ), we introduce the corresponding  $K$ -ring  $K(M, P)$ . In such a way we obtain the quantum Chern character, as a right  $K(M, P)$ -module structure  $\text{ch}_M: HZ(M) \times K(M, P) \rightarrow HZ(M)$ . Finally, we find an explicit expression for  $\text{ch}_M$ , in terms of the curvature map.

At the end of Section 4 we present a construction of quantum Chern classes, generalizing the classical “determinantal” definition. However, an interesting quantum phenomena appears—generally a given vector bundle will generate classes in arbitrary high dimensions.

In Section 5 properties of quantum  $U(1)$ -bundles are studied. A complete structural analysis of these bundles is performed, including the classification of differential calculi on them. Here the general formalism essentially simplifies, as far as we assume that the calculus on the structure group is 1-dimensional. In particular, this rules out the question about “embedded differential” maps, and uniquely fixes the “horizontal-vertical” decomposition. The algebra of characteristic classes is generated by a single element, the Euler class, given by the curvature form.

We shall also introduce the Euler class for certain vector bundles associated to general quantum principal bundles admitting regular connections and appropriate braided volume forms, following the classical analogy.

Section 6 is devoted to the study of characteristic classes of locally trivial bundles over classical smooth manifolds [5]. It turns out that characteristic classes of such a quantum principal bundle  $P$  can be viewed as standard characteristic classes of its classical part  $P_{cl}$  (the set of classical points of  $P$ ). We shall also describe an interesting class of line bundles which are not locally trivializable, and discuss their characteristic classes.

In Section 7 we present a construction of a natural differential calculus over quantum frame bundles. These structures generally give examples with non-trivial characteristic classes. However all computations can be performed completely because of the simple internal structure of the calculus.

Finally, in the last section some concluding observations and additional examples are collected. We end this introduction with several remarks concerning the relevant quantum group entities. Conceptually, we follow [13, 14] in a slightly different notation.

Let us consider a compact matrix quantum group  $G$ , represented by a Hopf  $*$ -algebra  $\mathcal{A}$ , interpreted as consisting of polynomial functions on  $G$ . Let  $\phi, \epsilon$  and  $\kappa$  be the coproduct, counit and the antipode map respectively. Let  $\text{ad}: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  be the adjoint action of  $G$  on itself.

Let  $\Gamma$  be a bicovariant  $*$ -calculus [14] over  $G$  and  $\Gamma^\wedge$  its universal differential envelope ([5]–Appendix B). Explicitly,

$$\Gamma^\wedge = \Gamma^\otimes / S^\wedge$$

where  $\Gamma^\otimes$  is the tensor bundle algebra, and  $S^\wedge \subseteq \Gamma^\otimes$  is the ideal generated by elements of the form

$$Q = \sum_k da_k db_k \quad \sum_k a_k db_k = 0.$$

The coproduct map  $\phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  is uniquely extendible to the homomorphism  $\widehat{\phi}: \Gamma^\wedge \rightarrow \Gamma^\wedge \widehat{\otimes} \Gamma^\wedge$  of graded-differential algebras. The restriction of this map on  $\Gamma$  is given by

$$\widehat{\phi}(\vartheta) = \ell_\Gamma(\vartheta) + \wp_\Gamma(\vartheta),$$

where  $\ell_\Gamma$  and  $\wp_\Gamma$  are the corresponding left and right action of  $G$  on  $\Gamma$ .

Let  $\Gamma_{inv}$  be the left-invariant part of  $\Gamma$ . We shall denote by  $\mathcal{R} \subseteq \ker(\epsilon)$  the right  $\mathcal{A}$ -ideal which determines  $\Gamma$ . There exists a natural surjection  $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$  given by

$$\pi(a) = \kappa(a^{(1)})da^{(2)}.$$

We have  $\ker(\pi) = \mathbb{C} \oplus \mathcal{R}$ , and hence there exists a natural isomorphism

$$\Gamma_{inv} \leftrightarrow \ker(\epsilon)/\mathcal{R}.$$

The product in  $\mathcal{A}$  induces a right  $\mathcal{A}$ -module structure on  $\Gamma_{inv}$ . We shall denote by  $\circ$  this structure. Explicitly,

$$\pi(a) \circ b = \pi(ab - \epsilon(a)b),$$

for each  $a, b \in \mathcal{A}$ .

Throughout this paper we shall deal with various bicovariant  $*$ -algebras  $\Gamma^*$  over  $\Gamma$ . The symbol  $\Gamma_{inv}^*$  will then refer to the corresponding left-invariant subalgebra. Here  $*$  symbolically characterizes the algebra—some other entities naturally related to  $\Gamma^*$  will be also endowed with the superscript  $*$ .

We have

$$\Gamma_{inv}^\wedge = \Gamma_{inv}^\otimes / S_{inv}^\wedge$$

where  $S_{inv}^\wedge$  is the left-invariant part of  $S^\wedge$ . As an ideal in  $\Gamma_{inv}^\otimes$ , it is generated by elements of the form

$$q = \pi(a^{(1)}) \otimes \pi(a^{(2)})$$

where  $a \in \mathcal{R}$ .

The  $\circ$  can be naturally extended from  $\Gamma_{inv}$  to  $\Gamma_{inv}^*$ , with the help of the formula

$$\vartheta \circ a = \kappa(a^{(1)})\vartheta a^{(2)},$$

so that we have

$$\begin{aligned} 1 \circ a &= \epsilon(a)1 \\ (\vartheta\eta) \circ a &= (\vartheta \circ a^{(1)})(\eta \circ a^{(2)}). \end{aligned}$$

The adjoint action  $\varpi: \Gamma_{inv} \rightarrow \Gamma_{inv} \otimes \mathcal{A}$  of  $G$  on  $\Gamma_{inv}$  is explicitly given by

$$\varpi\pi = (\pi \otimes \text{id})\text{ad},$$

it is naturally extendible to algebras  $\Gamma_{inv}^*$ .

Let  $\widehat{\varpi}: \Gamma_{inv}^\wedge \rightarrow \Gamma_{inv}^\wedge \widehat{\otimes} \Gamma^\wedge$  be the graded-differential extension of  $\varpi$ . We have

$$\widehat{\varpi}(\vartheta) = 1 \otimes \vartheta + \varpi(\vartheta),$$

for each  $\vartheta \in \Gamma_{inv}$ .

Finally, the following identities hold

$$\begin{aligned} d(\vartheta \circ a) &= d(\vartheta) \circ a - \pi(a^{(1)})(\vartheta \circ a^{(2)}) + (-1)^{\partial\vartheta}(\vartheta \circ a^{(1)})\pi(a^{(2)}) \\ d\pi(a) &= -\pi(a^{(1)})\pi(a^{(2)}) \\ (\vartheta \circ a)^* &= \vartheta^* \circ \kappa(a)^* \quad \pi(a)^* = -\pi[\kappa(a)^*]. \end{aligned}$$

All the above listed properties will be commonly used in various technical considerations.

## 2. GENERAL CONCEPTUAL FRAMEWORK

We pass to a brief exposition of general theory of quantum principal bundles, paying particular attention to differential calculus and the formalism of connections.

Let us consider a quantum space  $M$ , represented by a  $*$ -algebra  $\mathcal{V}$ . Let  $P = (\mathcal{B}, i, F)$  be a quantum principal  $G$ -bundle over  $M$ . Here  $\mathcal{B}$  is a  $*$ -algebra representing  $P$  as a quantum space, while  $i: \mathcal{V} \rightarrow \mathcal{B}$  and  $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$  are unital  $*$ -homomorphisms playing the role of the dualized projection of  $P$  on  $M$  and the right action of  $G$  on  $P$  respectively.

Let us assume that a differential calculus on the structure quantum group  $G$  is based on the universal envelope  $\Gamma^\wedge$  of a given first-order bicovariant  $*$ -calculus  $\Gamma$ . Let  $\Omega(P)$  be a graded-differential  $*$ -algebra representing the calculus on  $P$ . As first, this means that  $\Omega^0(P) = \mathcal{B}$ , and that  $\mathcal{B}$  generates the differential algebra  $\Omega(P)$ . Secondly the right action  $F$  is extendible (necessarily uniquely) to a graded-differential  $*$ -homomorphism  $\widehat{F}: \Omega(P) \rightarrow \Omega(P) \widehat{\otimes} \Gamma^\wedge$ .

The right action  $F^\wedge: \Omega(P) \rightarrow \Omega(P) \otimes \mathcal{A}$  of  $G$  on  $\Omega(P)$  is given by

$$F^\wedge = (\text{id} \otimes p_*)\widehat{F},$$

where  $p_*: \Gamma^\wedge \rightarrow \mathcal{A}$  is the projection map.

Let us consider a connection  $\omega: \Gamma_{inv} \rightarrow \Omega(P)$ . This means that  $\omega$  is a hermitian intertwiner between  $\varpi: \Gamma_{inv} \rightarrow \Gamma_{inv} \otimes \mathcal{A}$  and the right action  $F^\wedge: \Omega(P) \rightarrow \Omega(P) \otimes \mathcal{A}$ , satisfying

$$\pi_v \omega(\vartheta) = 1 \otimes \vartheta$$

where  $\pi_v: \Omega(P) \rightarrow \mathbf{ver}(P)$  is the verticalization homomorphism, and  $\mathbf{ver}(P)$  is the graded-differential  $*$ -algebra representing “verticalized” differential forms on the bundle (at the level of graded vector spaces,  $\mathbf{ver}(P) = \mathcal{B} \otimes \Gamma_{inv}^\wedge$ ). The bundle  $P$  possesses at least one connection. The set  $\mathbf{con}(P)$  of all connections on  $P$  is a real affine space, in a natural manner.

Connections can be equivalently described as hermitian first-order linear maps  $\omega: \Gamma_{inv} \rightarrow \Omega(P)$  satisfying

$$(2.1) \quad \widehat{F}\omega(\vartheta) = \sum_k \omega(\vartheta_k) \otimes c_k + 1 \otimes \vartheta,$$

where  $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$ .

The graded  $*$ -subalgebra of horizontal forms is defined by

$$\mathbf{hor}(P) = \widehat{F}^{-1} \left\{ \Omega(P) \otimes \mathcal{A} \right\}.$$

It turns out that  $\mathbf{hor}(P)$  is  $F^\wedge$ -invariant.

Let  $\Omega(M) \subseteq \Omega(P)$  be the graded-differential  $*$ -subalgebra representing differential forms on the base manifold  $M$ . The elements of  $\Omega(M)$  are characterized as  $\widehat{F}$ -invariant differential forms on  $P$ . Equivalently,  $\Omega(M)$  is a  $F^\wedge$ -fixed point subalgebra of  $\mathbf{hor}(P)$ . In some formulas we shall use a special symbol  $d_M: \Omega(M) \rightarrow \Omega(M)$  for the corresponding differential.

By definition,  $\omega$  is *regular* if the following commutation relation holds

$$(2.2) \quad \omega(\vartheta)\varphi = (-1)^{\partial\varphi} \sum_k \varphi_k \omega(\vartheta \circ c_k)$$

where  $\varphi$  is an arbitrary horizontal form, and  $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$ . In particular, regular connections graded-commute with the elements of  $\Omega(M)$ .

A connection  $\omega$  is called *multiplicative* iff it is extendible (necessarily uniquely) to a (unital)  $*$ -homomorphism  $\omega^\wedge: \Gamma_{inv}^\wedge \rightarrow \Omega(P)$ .

It is important to observe that multiplicativity and regularity properties essentially depend on the complete differential calculus on the bundle. In general,  $\Omega(P)$  may be such that there will be no regular nor multiplicative connections. Furthermore, regular connections, if exist, are not necessarily multiplicative. However, if one regular connection is multiplicative, then all regular connections will be multiplicative, too. If regular connections are not multiplicative then it is possible to “renormalize” the calculus on the bundle, by factorizing  $\Omega(P)$  through an appropriate graded-differential  $*$ -ideal, measuring a lack of multiplicativity of regular connections. Such a factorization does not change the first-order calculus. In terms of the renormalized calculus, regular connections are multiplicative.

The map  $\omega^\wedge: \Gamma_{inv}^\wedge \rightarrow \Omega(P)$  can be introduced for an arbitrary connection, by the formula

$$\omega^\wedge = \omega^\otimes \iota$$

where  $\iota: \Gamma_{inv}^\wedge \rightarrow \Gamma_{inv}^\otimes$  is an arbitrary grade-preserving hermitian section of the factorization map intertwining the adjoint actions of  $G$ , and  $\omega^\otimes: \Gamma_{inv}^\otimes \rightarrow \Omega(P)$  is the unital multiplicative extension of  $\omega$ .

The quantum counterparts of horizontal projection, covariant derivative and curvature can be constructed as follows. Let  $\mathbf{vh}(P)$  be the graded  $*$ -algebra of

“vertically-horizontally” decomposed forms. Explicitly  $\mathfrak{vh}(P) = \mathfrak{hor}(P) \otimes \Gamma_{inv}^\wedge$ , at the level of graded vector spaces. The  $*$ -algebra structure is specified by

$$\begin{aligned} (\psi \otimes \eta)(\varphi \otimes \vartheta) &= (-1)^{\partial\eta\partial\varphi} \sum_k \psi \varphi_k \otimes (\eta \circ c_k) \vartheta \\ (\varphi \otimes \vartheta)^* &= \sum_k \varphi_k^* \otimes (\vartheta^* \circ c_k^*). \end{aligned}$$

Let us consider the “decomposition map”  $m_\omega: \mathfrak{vh}(P) \rightarrow \Omega(P)$ , given by

$$m_\omega(\varphi \otimes \vartheta) = \varphi \omega^\wedge(\vartheta).$$

This map is bijective, and intertwines the corresponding right actions of  $G$ . Moreover, if  $\omega$  is regular and multiplicative then  $m_\omega$  is a  $*$ -algebra isomorphism.

The horizontal projection operator  $h_\omega: \Omega(P) \rightarrow \mathfrak{hor}(P)$  is given by

$$h_\omega = (\text{id} \otimes p_*) m_\omega^{-1}.$$

In a direct analogy with classical differential geometry, the covariant derivative operator can be defined by

$$D_\omega = h_\omega d.$$

The operators  $D_\omega$  and  $h_\omega$  are right-covariant, in a natural manner. Finally, the curvature  $R_\omega: \Gamma_{inv} \rightarrow \Omega(P)$  can be defined as the composition

$$R_\omega = D_\omega \omega.$$

This is a hermitian tensorial 2-form. The following identity holds

$$R_\omega = d\omega - \langle \omega, \omega \rangle.$$

Here  $\langle \rangle$  are the brackets associated to the embedded differential  $\delta: \Gamma_{inv} \rightarrow \Gamma_{inv}^{\otimes 2}$ , where  $\delta(\vartheta) = \iota d(\vartheta)$ . The above identity generalizes the classical Structure equation.

It is worth noticing that  $R_\omega$  depends, besides on  $\omega$ , also on the map  $\delta$ . In a large part of the formalism, the section  $\iota$  figures only via the corresponding  $\delta$ . The map  $\delta$  is always related with the transposed commutator [14] map  $c^\top = (\text{id} \otimes \pi)\varpi$  in the following way

$$(2.3) \quad \sigma\delta = \delta + c^\top,$$

where  $\sigma: \Gamma_{inv}^{\otimes 2} \rightarrow \Gamma_{inv}^{\otimes 2}$  is the (left-invariant restriction of the) canonical braid operator [14]. It is explicitly given by

$$(2.4) \quad \sigma(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k),$$

where  $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$ .

The map  $c^\top$  is extendible to  $c^\top: \Gamma_{inv}^* \rightarrow \Gamma_{inv}^* \otimes \Gamma_{inv}$ , by the same formula. The elements of  $S_{inv}^{\wedge 2}$  are  $\sigma$ -invariant, and hence  $\sigma$  projects to  $\sigma: \Gamma_{inv}^{\wedge 2} \rightarrow \Gamma_{inv}^{\wedge 2}$ .

Let  $I^* \subseteq \Gamma_{inv}^{\otimes}$  be the graded  $*$ -subalgebra consisting of elements invariant under the adjoint action  $\varpi: \Gamma_{inv}^{\otimes} \rightarrow \Gamma_{inv}^{\otimes} \otimes \mathcal{A}$ . The formula

$$(2.5) \quad W^\omega(\vartheta) = R_\omega^\otimes(\vartheta)$$

defines a  $*$ -homomorphism  $W^\omega: I^* \rightarrow \Omega(M)$ . Here  $R_\omega^\otimes: \Gamma_{inv}^\otimes \rightarrow \Omega(P)$  is the corresponding unital multiplicative extension.

We shall denote by  $Z(M)$  the graded centre of  $\Omega(M)$ . It is a graded-differential  $*$ -subalgebra of  $\Omega(M)$ .

If  $\omega$  is regular then the following identity holds

$$R_\omega(\vartheta)\varphi = \sum_k \varphi_k R_\omega(\vartheta \circ c_k),$$

for each  $\varphi \in \mathfrak{hor}(P)$ . In particular, the curvature  $R_\omega$  commutes with all elements of  $\Omega(M)$ , and hence the image of  $W^\omega$  is contained in  $Z(M)$ .

For the end of this section, we consider the internal structure of the algebras  $\mathcal{B}$  and  $\mathfrak{hor}(P)$ , from the point of view of the representation theory [14] of compact matrix quantum groups. The symbol  $\otimes_M$  will be used for both tensor products over  $\mathcal{V}$  and  $\Omega(M)$ , depending on the context.

Let  $R(G)$  be a complete monoidal category of finite-dimensional unitary representations of  $G$ . Each  $u \in R(G)$  is understandable as a map  $u: H_u \rightarrow H_u \otimes \mathcal{A}$ , where  $H_u$  is the corresponding carrier unitary space (we shall assume that  $H_u$  belong to a fixed model set of unitary spaces). Morphisms of the category  $R(G)$  are intertwiners between representations. We shall use the symbols  $\oplus, \times$  for the sum and the product in  $R(G)$ . We have

$$H_{u \oplus v} = H_u \oplus H_v \quad H_{u \times v} = H_u \otimes H_v.$$

The symbol  $\emptyset$  will refer to the *trivial* representation of  $G$  acting in the space  $H_\emptyset = \mathbb{C}$ .

For each  $u \in R(G)$  let  $\mathcal{E}_u = \text{Mor}(u, F)$  be the space of intertwiners between  $u$  and  $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ . The spaces  $\mathcal{E}_u$  are  $\mathcal{V}$ -bimodules, in a natural manner. Furthermore  $\mathcal{E}_\emptyset = \mathcal{V}$ , in a natural manner.

For each  $u, v \in R(G)$ , the following natural bimodule isomorphism holds:

$$(2.6) \quad \mathcal{E}_{u \times v} \leftrightarrow \mathcal{E}_u \otimes_M \mathcal{E}_v.$$

The above isomorphism is induced by the product in  $\mathcal{B}$ . Explicitly

$$\varphi \otimes \psi: x \otimes y \mapsto \varphi(x)\psi(y).$$

Let us denote by  $u^c = \bar{u}$  the contragradient representation, identified with the conjugate representation, acting in  $H_{u^c} = H_u^*$ . Let  $C_u: H_u \rightarrow H_u$  be the canonical intertwiner between  $u$  and the second contragradient  $u^{cc}$ . This map is strictly positive, and normalized such that  $\text{tr}(C_u) = \text{tr}(C_u^{-1})$ . Furthermore,

$$C_u \otimes C_v = C_{u \times v} \quad \varphi C_u = C_v \varphi$$

for each  $u, v \in R(G)$  and  $\varphi \in \text{Mor}(u, v)$ . The appropriate scalar product in  $H_u^*$  is given by

$$(f, g) = (j_u^{-1}(g), C_u j_u^{-1}(f))$$

where  $j_u: H_u \rightarrow H_u^*$  is the antilinear map induced by the scalar product in  $H_u$ .



There exists a natural bimodule anti-isomorphism  $*_u: \mathcal{E}_u \rightarrow \mathcal{E}_{\bar{u}}$ , determined by the diagram

$$(2.7) \quad \begin{array}{ccc} H_u & \xrightarrow{\varphi} & \mathcal{B} \\ j_u \downarrow & & \downarrow * \\ H_u^* & \xrightarrow{*_u \varphi} & \mathcal{B} \end{array}$$

In other words, we can naturally identify  $\mathcal{E}_{\bar{u}}$  and the corresponding conjugate bimodule  $\mathcal{E}_u^*$ .

Let  $\text{rhom}(\mathcal{E}_u, \mathcal{E}_v)$  be the space of right  $\mathcal{V}$ -linear maps  $A: \mathcal{E}_u \rightarrow \mathcal{E}_v$ . This is a  $\mathcal{V}$ -bimodule, in a natural manner. Similarly, we shall denote by  $\text{lhom}(\mathcal{E}_u, \mathcal{E}_v)$  the space of corresponding left  $\mathcal{V}$ -module homomorphisms. Finally, let us denote by  $\text{hom} = \text{lhom} \cap \text{rhom}$  the spaces of bimodule homomorphisms. For  $u = v$  all introduced spaces are algebras.

Each map  $f \in \text{Mor}(u, v)$  induces a bimodule homomorphism  $f_*: \mathcal{E}_v \rightarrow \mathcal{E}_u$ , given by the composition of intertwiners.

From the point of view of our considerations, the most interesting intertwiners are given by the standard contraction map  $\gamma^u: H_u^* \otimes H_u \rightarrow \mathbb{C}$ , and the natural inclusion  $I_u: \mathbb{C} \rightarrow H_u \otimes H_u^* = L(H_u)$ . These intertwiners induce bimodule homomorphisms  $\gamma_*^u: \mathcal{V} \rightarrow \mathcal{E}_u^* \otimes_M \mathcal{E}_u$ , and  $\langle \rangle_u^+: \mathcal{E}_u \otimes_M \mathcal{E}_u^* \rightarrow \mathcal{V}$  respectively. The map  $\gamma_*^u$  is completely determined by

$$\gamma_*^u(1) = \sum_k \nu_k \otimes \mu_k.$$

By definition, the following identity holds

$$(2.8) \quad \sum_k \nu_k(e_i^*) \mu_k(e_j) = \delta_{ij} 1,$$

where  $\{e_i\}$  is an arbitrary orthonormal basis in  $H_u$  and  $\{e_i^*\}$  is the corresponding biorthogonal basis in  $H_u^*$ .

The map  $\langle \rangle_u^+$  is naturally understandable as a pairing between  $\mathcal{E}_u$  and  $\mathcal{E}_u^*$ . Explicitly,

$$(2.9) \quad \langle \varphi, \psi \rangle_u^+ = \sum_i \varphi(e_i) \psi(e_i^*).$$

Similarly, it is possible to reverse the roles of  $u$  and  $\bar{u}$ . In particular, we can construct the canonical pairing  $\langle \rangle_u^-: \mathcal{E}_u^* \otimes_M \mathcal{E}_u \rightarrow \mathcal{V}$ . It is given by

$$(2.10) \quad \langle \psi, \varphi \rangle_u^- = \sum_{ij} (C_u^{-1})_{ji} \psi(e_i^*) \varphi(e_j).$$

The operator  $C_u^{-1}$  in the above expression comes from the compensation of the appearance of the second conjugate  $u^{cc}$  in place of  $u$ .

The structure of the bundle  $P$  is expressible [7] in terms of the system of bimodules  $\mathcal{E}_u$ , together with maps of the form  $*_u$  and  $f_*$ . In particular, the following

natural decomposition holds

$$(2.11) \quad \mathcal{B} = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathcal{B}^{\alpha},$$

where  $\mathcal{T}$  is the complete set of mutually inequivalent irreducible unitary representations of  $G$ , and  $\mathcal{B}^{\alpha}$  are multiple irreducible  $\mathcal{V}$ -submodules with respect to the decomposition of  $F$ . These modules are further naturally decomposable as follows

$$(2.12) \quad \mathcal{B}^{\alpha} = \mathcal{E}_{\alpha} \otimes H_{\alpha} \quad \varphi(x) \leftrightarrow \varphi \otimes x.$$

A similar consideration can be applied to the algebra  $\mathfrak{hor}(P)$ . This leads to the intertwiner  $\Omega(M)$ -bimodules  $\mathcal{F}_u = \text{Mor}(u, F^{\wedge})$ . We shall use the same symbols for the characteristic spaces and operators related to these bimodules. The spaces  $\mathcal{F}_u$  are naturally graded, and obviously  $\mathcal{F}_u^0 = \mathcal{E}_u$ . Moreover, it turns out that right/left product maps in  $\mathcal{F}_u$  induce canonical decompositions

$$(2.13) \quad \mathcal{E}_u \otimes_M \Omega(M) \leftrightarrow \mathcal{F}_u \leftrightarrow \Omega(M) \otimes_M \mathcal{E}_u.$$

In particular  $\text{lhom}(\mathcal{E}_u, \mathcal{E}_v)$  is realized as a subalgebra of  $\text{lhom}(\mathcal{F}_u, \mathcal{F}_v)$ , consisting of grade-preserving transformations.

The structure of spaces  $\mathcal{F}_u$  can be expressed in terms of spaces  $\mathcal{E}_u$ . Composing the above two identifications we obtain canonical flip-over maps

$$\sigma_u: \mathcal{E}_u \otimes_M \Omega(M) \rightarrow \Omega(M) \otimes_M \mathcal{E}_u.$$

By definition, these maps are grade-preserving and act as the identity on  $\mathcal{E}_u$ . The  $\Omega(M)$ -bimodule structure on  $\mathcal{F}_u$  is expressed by the following compatibility condition

$$(2.14) \quad \sigma_u(\text{id} \otimes m_M) = (m_M \otimes \text{id})(\text{id} \otimes \sigma_u)(\sigma_u \otimes \text{id}),$$

where  $m_M$  is the product in  $\Omega(M)$ . Intertwiner homomorphisms  $f_{\star}$  and conjugation maps  $*_u$  satisfy the following diagrams

$$\begin{array}{ccc} \mathcal{E}_v \otimes_M \Omega(M) & \xrightarrow{\sigma_v} & \Omega(M) \otimes_M \mathcal{E}_v \\ f_{\star} \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes f_{\star} \\ \mathcal{E}_u \otimes_M \Omega(M) & \xrightarrow{\sigma_u} & \Omega(M) \otimes_M \mathcal{E}_u \end{array} \quad \begin{array}{ccc} \mathcal{E}_u \otimes_M \Omega(M) & \xrightarrow{\sigma_u} & \Omega(M) \otimes_M \mathcal{E}_u \\ *_u \downarrow & & \downarrow *_u \\ \Omega(M) \otimes_M \mathcal{E}_{\bar{u}} & \xleftarrow{\sigma_{\bar{u}}} & \mathcal{E}_{\bar{u}} \otimes_M \Omega(M) \end{array}$$

The first diagram actually characterises elements of  $\text{hom}(\mathcal{E}_v, \mathcal{E}_u)$  that are extendible to corresponding  $\Omega(M)$ -bimodule homomorphisms (the left and right  $\Omega(M)$ -linear extensions coincide).

The algebra of horizontal forms can be decomposed in the following way

$$(2.15) \quad \begin{aligned} \mathfrak{hor}(P) &= \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathcal{H}^{\alpha}(P) & \mathcal{H}^{\alpha}(P) &= \mathcal{F}_{\alpha} \otimes H_{\alpha} \\ \mathfrak{hor}(P) &\leftrightarrow \Omega(M) \otimes_M \mathcal{B} \leftrightarrow \mathcal{B} \otimes_M \Omega(M). \end{aligned}$$

Let us observe that the elements of  $\text{lhom}(\mathcal{F}_u, \mathcal{F}_v)$  are also naturally interpretable as maps  $A$  satisfying

$$(2.16) \quad A(\alpha\psi) = (-1)^{\partial\alpha\partial A} \alpha A(\psi).$$

In what follows we shall commonly pass from one interpretation to another. We shall denote by  $\text{ghom}(\mathcal{F}_u, \mathcal{F}_v)$  the space of elements  $A \in \text{rhom}(\mathcal{F}_u, \mathcal{F}_v)$  satisfying also (2.16).

Let us now consider a regular connection  $\omega$  on  $P$ . The covariant derivative map  $D_\omega$  is a right-covariant hermitian first-order derivation on  $\mathfrak{hor}(P)$ . The composition of  $D_\omega$  with elements of  $\mathcal{F}_u$  induces first-order maps  $D_{\omega,u}: \mathcal{F}_u \rightarrow \mathcal{F}_u$ .

The graded Leibniz rule for  $D_\omega$  and the fact that  $(D_\omega - d_M)(\Omega(M)) = \{0\}$  imply

$$D_{\omega,u \times v}(\varphi \otimes \psi) = D_{\omega,u}(\varphi) \otimes \psi + (-1)^{\partial \varphi} \varphi \otimes D_{\omega,v}(\psi) \quad D_{\omega,\emptyset} = d_M.$$

Moreover, the following compatibility properties hold

$$D_{\omega,\bar{u}} * u = * u D_{\omega,u} \quad D_{\omega,u} f_* = f_* D_{\omega,v}.$$

### 3. QUANTUM CHARACTERISTIC CLASSES

In this section the abstract theory of characteristic classes will be presented. We shall start from a particular case, when the bundle admits regular connections (and the calculus is “renormalized” such that regular connections are multiplicative). Then the construction of the Weil homomorphism can be performed in a direct analogy with the classical case. Then we pass to a more general geometrical framework, which however still allows to perform the constructions as in the regular case. Finally, we present a general cohomological approach to characteristic classes, in the framework of which “universal” characteristic classes are defined. We shall consider separately the general and the regular case. These universal classes naturally form the domain of the Weil homomorphism. A particular attention will be payed to the study of interrelations between two levels of generality. The formalism is applicable to arbitrary bundles.

#### 3.1. The Regular Case

Let us assume that the bundle admits regular (and multiplicative) connections. Then it is possible to prove, straightforwardly generalizing the classical exposition [11], that  $dW^\omega(\vartheta) = 0$  for each  $\vartheta \in I^*$  and  $\omega \in \mathfrak{r}(P)$ . Moreover [6], the cohomological class of  $W^\omega(\vartheta)$  in  $\Omega(M)$  is independent of the choice of  $\omega \in \mathfrak{r}(P)$ .

**Lemma 3.1.** *Actually a finer result holds—the above cohomological class is well defined in  $Z(M)$ .  $\square$*

In such a way we have constructed an intrinsic  $*$ -homomorphism

$$W: I^* \rightarrow HZ(M) \quad W(\vartheta) = [W^\omega(\vartheta)],$$

where  $HZ(M)$  is the corresponding cohomology algebra. This map is a quantum counterpart of the Weil homomorphism. We shall denote by  $\mathfrak{w}(M) \subseteq HZ(M)$  the image of  $W$ .

The quantum Weil homomorphism can be further factorized through the ideal  $J_\sigma$  generated by the space  $\text{im}(I - \sigma) \subseteq \Gamma_{inv}^{\otimes 2}$ . This follows from the commutation identity

$$R_\omega(\eta)R_\omega(\vartheta) = \sum_k R_\omega(\vartheta_k)R_\omega(\eta \circ c_k)$$

where  $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$ . The elements of the factoralgebra  $\Sigma = \Gamma_{inv}^\otimes / J_\sigma$  are interpretable as “polynoms” over the “Lie algebra” of  $G$ . The adjoint action  $\varpi^\otimes$  is naturally projectable on  $\Sigma$ . Let  $I(\Sigma) \subseteq \Sigma$  be the subalgebra of elements invariant under the projected action (invariant polynomials). We have  $I(\Sigma) = I^* / (I^* \cap J_\sigma)$ .

From the commutation relations defining the algebra  $\Sigma$  it follows that  $I(\Sigma)$  is a *central* subalgebra of  $\Sigma$ . In particular, it is commutative. The algebra  $I(\Sigma)$  is a natural domain for the Weil homomorphism, in the regular case.

### 3.2. Some Variations

The construction described in the previous subsection can be incorporated into a more general context, including some situations when the (calculus on the) bundle does not admit regular connections.

We shall first introduce a sufficient condition for the calculus, which allows that the construction of the Weil homomorphism works essentially in the same way as in the regular case.

Let us consider a quantum principal bundle  $P$ , endowed with a differential calculus  $\Omega(P)$ . For each  $\omega \in \text{con}(P)$  let  $\mathcal{K}_\omega \subseteq \Omega(P)$  be an ideal generated by elements of the form

$$(3.1) \quad v_{\varphi, \omega} = \langle \omega, \varphi \rangle - (-1)^{\partial \varphi} \langle \varphi, \omega \rangle,$$

where  $\varphi$  is an arbitrary pseudotensorial form of the type  $\varpi$ . Evidently,  $\mathcal{K}_\omega$  is  $*$ -invariant.

Let us assume that the embedded differential map  $\delta$  is such that there exists a real affine space  $\mathcal{X}$  consisting of connections  $\omega$  satisfying

$$(3.2) \quad \mathfrak{hor}(P) \cap \mathcal{K}_\omega = \{0\}.$$

Evidently, if the bundle admits regular (and multiplicative) connections then we can choose  $\mathcal{X} = \mathfrak{t}(P)$ . However, condition (3.2) is much weaker than the regularity assumption. In particular, if the embedded differential  $\delta$  vanishes identically then  $\mathcal{K}_\omega = \{0\}$  and (3.2) holds automatically. This covers quantum line bundles, for example.

**Proposition 3.2.** (i) *If  $\omega \in \mathcal{X}$  then  $W^\omega(\vartheta)$  is closed, for each  $\vartheta \in I^*$ .*

(ii) *The cohomological class of  $W^\omega(\vartheta)$  in  $\Omega(M)$  is independent of the choice of  $\omega \in \mathcal{X}$ , and a map  $W: I^* \rightarrow H(M)$  is a  $*$ -homomorphism.*

*Proof.* Although the proof is similar as in regular case [6], we include it here for completeness. Applying the structure equation we find

$$(d\omega)^\otimes(\vartheta) = W^\omega(\vartheta) + \sum_{\alpha\beta\eta} \alpha \langle \omega, \omega \rangle(\eta) \beta$$

for some  $\alpha, \beta \in \Omega(P)$  and  $\eta \in \Gamma_{inv}$ . Differentiating both sides we obtain

$$\begin{aligned} dW^\omega(\vartheta) = & - \sum_{\alpha\beta\eta} \left\{ d\alpha \langle \omega, \omega \rangle(\eta) \beta + (-1)^{\partial \alpha} \alpha \langle \omega, \omega \rangle(\eta) d\beta \right\} \\ & - \sum_{\alpha\beta\eta} (-1)^{\partial \alpha} \alpha \left[ \langle d\omega, \omega \rangle(\eta) - \langle \omega, d\omega \rangle(\eta) \right] \beta. \end{aligned}$$

If  $\omega \in \mathcal{X}$  then the above equality is possible only if  $dW^\omega(\vartheta) = 0$ , which proves the first statement. Let us consider another connection  $\tau \in \mathcal{X}$ , and let  $\omega_t = \omega + t(\tau - \omega)$  be the segment in  $\mathcal{X}$  determined by  $\omega$  and  $\tau$ . We have then

$$\frac{d}{dt}W^{\omega_t}(\vartheta) = dw(\vartheta) + \sum_{fg\eta} f(\langle\varphi, \omega_t\rangle + \langle\omega_t, \varphi\rangle)(\eta)g + \sum_{uv\zeta} u(\langle d\omega_t, \omega_t\rangle - \langle\omega_t, d\omega_t\rangle)(\zeta)v,$$

where  $w(\vartheta) \in \Omega(M)$  and  $f, g \in \mathfrak{hor}(P)$  are expressed in terms of  $R_{\omega_t}$ , while  $\varphi = \omega - \tau$ . The splitting assumption (3.2) implies that the sum of  $\mathcal{K}_\omega$ -terms vanishes. Integrating the above equality from 0 to 1 we conclude that  $W^\omega$  and  $W^\tau$  coincide in  $\Omega(M)$  up to a coboundary.  $\square$

In contrast to the regular case however, the Weil homomorphism is generally not factorizable to the corresponding symmetric algebra. Another important difference between the regular and the described case is that the Weil homomorphism  $W$  is not longer  $HZ(M)$ -valued.

In the context of general bundles and connections on them, the covariant derivative is not hermitian, and does not satisfy the graded Leibniz rule. However, appropriate generalizations of these properties hold in the general case. A deviation of regularity of an arbitrary connection  $\omega$  is naturally expressed via the operator  $\ell_\omega: \Gamma_{inv} \times \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$  defined by

$$(3.3) \quad \ell_\omega(\vartheta, \varphi) = \omega(\vartheta)\varphi - (-1)^{\partial\varphi} \sum_k \varphi_k \omega(\vartheta \circ c_k).$$

With the help of this operator, it is possible to formulate “hermicity” and the “Leibniz rule” for an arbitrary covariant derivative.

**Proposition 3.3.** (i) *The following identities hold*

$$\begin{aligned} D_\omega(\varphi\psi) &= D_\omega(\varphi)\psi + (-1)^{\partial\varphi} \varphi D_\omega(\psi) + (-1)^{\partial\varphi} \sum_k \varphi_k \ell_\omega(\pi(c_k), \psi) \\ D_\omega(\varphi)^* &= D_\omega(\varphi^*) + \sum_k \ell_\omega(\pi\kappa(c_k)^*, \varphi_k^*), \end{aligned}$$

where  $\varphi, \psi \in \mathfrak{hor}(P)$ .

(ii) *Let us consider a map  $\rho: \Gamma_{inv} \otimes \mathcal{A} \rightarrow S_{inv}^{\wedge 2}$  given by*

$$\rho(\vartheta, a) = \delta(\vartheta) \circ a - \delta(\vartheta \circ a) - (\vartheta \circ a^{(1)}) \otimes \pi(a^{(2)}) - \pi(a^{(1)}) \otimes (\vartheta \circ a^{(2)}).$$

*Let  $u_\omega: \Gamma_{inv} \otimes \mathcal{A} \rightarrow \mathfrak{hor}(P)$  be a linear map obtained by composing  $\omega^\otimes$  and  $\rho$ . Then the following identity holds*

$$(3.4) \quad \begin{aligned} D_\omega \ell_\omega(\vartheta, \varphi) &= R_\omega(\vartheta)\varphi - \sum_k \varphi_k R_\omega(\vartheta \circ c_k) - \ell_\omega(\vartheta, D_\omega(\varphi)) \\ &\quad + \sum_k \varphi_k u_\omega(\vartheta, c_k) + \sum_j \ell_\omega(\vartheta_j^1, \ell_\omega(\vartheta_j^2, \varphi)), \end{aligned}$$

where  $\delta(\vartheta) = \sum_j \vartheta_j^1 \otimes \vartheta_j^2$ .

*Proof.* Here we shall prove property (ii). The statement (i) follows by modifying trivially a proof given in [6]. A direct computation gives

$$\begin{aligned}
\langle \omega, \omega \rangle(\vartheta) \varphi &= \sum_j \omega(\vartheta_j^1) \ell_\omega(\vartheta_j^2, \varphi) + (-1)^{\partial \varphi} \sum_{kj} \omega(\vartheta_j^1) \varphi_k \omega(\vartheta_j^2 \circ c_k) \\
&= \sum_j \ell_\omega(\vartheta_j^1, \ell_\omega(\vartheta_j^2, \varphi)) - (-1)^{\partial \varphi} \sum_{jk} \ell_\omega(\vartheta_j^1, \varphi_k) \omega(\vartheta_j^2 \circ c_k) \\
&\quad + (-1)^{\partial \varphi} \sum_{kj} \ell_\omega(\vartheta_j^1, \varphi_k) \omega(\vartheta_j^2 \circ c_k) + \sum_{kj} \varphi_k \omega(\vartheta_j^1 \circ c_k^{(1)}) \omega(\vartheta_j^2 \circ c_k^{(2)}) \\
&= \sum_j \ell_\omega(\vartheta_j^1, \ell_\omega(\vartheta_j^2, \varphi)) - (-1)^{\partial \varphi} \sum_{kl} \ell_\omega(\vartheta_l, \varphi_k) \omega[\pi(d_l) \circ c_k] \\
&\quad + \sum_{kj} \varphi_k \omega(\vartheta_j^1 \circ c_k^{(1)}) \omega(\vartheta_j^2 \circ c_k^{(2)}),
\end{aligned}$$

where  $\sum_j \vartheta_j^1 \otimes \vartheta_j^2 = \sigma \delta(\vartheta)$  and  $\varpi(\vartheta) = \sum_j \vartheta_j \otimes d_j$ . On the other hand

$$\begin{aligned}
D_\omega \ell_\omega(\vartheta, \varphi) &= d\ell_\omega(\vartheta, \varphi) + (-1)^{\partial \varphi} \sum_{jk} \ell_\omega(\vartheta_j, \varphi_k) \omega \pi(d_j c_k) \\
&= d\omega(\vartheta) \varphi - \omega(\vartheta) d\varphi - (-1)^{\partial \varphi} \sum_k d(\varphi_k) \omega(\vartheta \circ c_k) - \sum_k \varphi_k d\omega(\vartheta \circ c_k) \\
&\quad + (-1)^{\partial \varphi} \sum_{kj} \omega(\vartheta_j) \varphi_k \omega(\pi(d_j) \circ c_k) + (-1)^{\partial \varphi} \sum_k \omega(\vartheta) \varphi_k \omega \pi(c_k) \\
&\quad - \sum_{kj} \varphi_k \omega(\vartheta_j \circ c_k^{(1)}) \omega(\pi(d_j) \circ c_k^{(2)}) - \sum_k \varphi_k \omega(\vartheta \circ c_k^{(1)}) \omega \pi(c_k^{(2)}) \\
&= d\omega(\vartheta) \varphi - \omega(\vartheta) D_\omega(\varphi) - (-1)^{\partial \varphi} \sum_k D_\omega(\varphi_k) \omega(\vartheta \circ c_k) \\
&\quad - \sum_k \varphi_k \omega(\vartheta \circ c_k^{(1)}) \omega \pi(c_k^{(2)}) - \sum_k \varphi_k \omega \pi(c_k^{(1)}) \omega(\vartheta \circ c_k^{(2)}) \\
&\quad + (-1)^{\partial \varphi} \sum_{kj} \ell_\omega(\vartheta_j, \varphi_k) \omega(\pi(d_j) \circ c_k) - \sum_k \varphi_k d\omega(\vartheta \circ c_k) \\
&= R_\omega(\vartheta) \varphi + \langle \omega, \omega \rangle(\vartheta) \varphi - \sum_k \varphi_k R_\omega(\vartheta \circ c_k) - \ell_\omega(\vartheta, D_\omega(\varphi)) \\
&\quad + (-1)^{\partial \varphi} \sum_{jk} \ell_\omega(\vartheta_j, \varphi_k) \omega(\pi(d_j) \circ c_k) - \sum_k \varphi_k \langle \omega, \omega \rangle(\vartheta \circ c_k) \\
&\quad - \sum_k \varphi_k \omega(\vartheta \circ c_k^{(1)}) \omega \pi(c_k^{(2)}) - \sum_k \varphi_k \omega \pi(c_k^{(1)}) \omega(\vartheta \circ c_k^{(2)}).
\end{aligned}$$

Combining the above expressions, and performing further elementary transformations we conclude that (3.4) holds.  $\square$

In general case, for an arbitrary bundle  $P$  and a connection  $\omega$  on  $P$ , it is possible to associate to a given calculus  $\Omega(P)$  a new calculus, by passing to an appropriate

factoralgebra, such that the “projected”  $\omega$  is regular. Let  $\mathcal{N}_\omega \subseteq \mathfrak{hor}(P)$  be a two-sided ideal generated by elements of the form  $\ell_\omega(\vartheta, \varphi)$  and

$$n_\omega(\vartheta, \varphi) = R_\omega(\vartheta)\varphi - \sum_k \varphi_k R_\omega(\vartheta \circ c_k) + \sum_k \varphi_k u_\omega(\vartheta, c_k),$$

where  $\vartheta \in \Gamma_{inv}$  and  $\varphi \in \mathfrak{hor}(P)$ .

**Proposition 3.4.** *The following properties hold*

$$F^\wedge(\mathcal{N}_\omega) \subseteq \mathcal{N}_\omega \otimes \mathcal{A} \quad *(\mathcal{N}_\omega) = \mathcal{N}_\omega \quad D_\omega(\mathcal{N}_\omega) \subseteq \mathcal{N}_\omega.$$

*Proof.* A direct calculation shows that

$$\begin{aligned} \rho(\vartheta, a) \circ b &= \rho(\vartheta, ab) - \rho(\vartheta \circ a, b) \\ \rho(\vartheta, a)^* &= \rho(\vartheta^*, \kappa(a)^*) \\ \varpi\rho(\vartheta, a) &= \sum_l \rho(\vartheta_l, a^{(2)}) \otimes \kappa(a^{(1)}) d_l a^{(3)}. \end{aligned}$$

It follows that

$$\begin{aligned} n_\omega(\vartheta, \varphi)^* &= - \sum_k n_\omega(\vartheta^* \circ \kappa(c_k)^*, \varphi_k^*) + \{\ell_\omega\text{-terms}\} \\ F^\wedge n_\omega(\vartheta, \varphi) &= \sum_{kl} n_\omega(\vartheta_l, \varphi_k) \otimes d_l c_k. \end{aligned}$$

This, together with elementary transformation properties of  $\ell_\omega$  shows that  $\mathcal{N}_\omega$  is  $*, F^\wedge$ -invariant.

The  $D_\omega$ -invariance of  $\mathcal{N}_\omega$  follows from the fact that  $D_\omega$  satisfies graded Leibniz rule modulo  $\ell_\omega$ -terms, the formula (3.4), as well as from the formula for the square [6] of  $D_\omega$ . The space  $\mathcal{N}_\omega$  can be characterized as the minimal  $D_\omega$ -invariant ideal in  $\mathfrak{hor}(P)$  containing all the elements  $\ell_\omega(\vartheta, \varphi)$ .  $\square$

Let us consider the factoralgebra

$$\mathfrak{h}(P) = \mathfrak{hor}(P)/\mathcal{N}_\omega.$$

The  $*$ -structure and the right action  $F^\wedge$  are projectable from  $\mathfrak{hor}(P)$  to  $\mathfrak{h}(P)$ . Moreover, the covariant derivative  $D_\omega$  projects to a hermitian  $F^\wedge$ -covariant antiderivation  $D: \mathfrak{h}(P) \rightarrow \mathfrak{h}(P)$ . Now, we can apply to  $\{\mathfrak{h}, F^\wedge, *, D\}$  a construction of differential calculus presented in [8]. In such a way we obtain a graded-differential  $*$ -algebra  $\Omega_P$ , representing a new calculus on the bundle, a bicovariant  $*$ -calculus  $\Psi$  over  $G$ , and a regular (and multiplicative) connection  $\tau: \Psi_{inv} \rightarrow \Omega_P$  such that  $D_\tau = D$ . However, if the initial connection  $\omega$  is sufficiently “irregular” then the ideal  $\mathcal{N}_\omega$  will be inappropriately large.

### 3.3. The Spectral Sequence

We pass to the study of a spectral sequence naturally associated to every quantum principal bundle  $P$ , endowed with a differential structure  $\Omega(P)$ . The sequence is based on a filtration of  $\Omega(P)$ , induced by the pull back map  $\widehat{F}$ . The whole analysis can be viewed as a variation of a classical theme, presented in [3] for example.

For each  $k \geq 0$  let  $\Omega_k(P) \subseteq \Omega(P)$  be the space consisting of elements having the “vertical order” less or equal  $k$ . In other words,

$$\Omega_k(P) = \widehat{F}^{-1}(\Omega(P) \otimes \Gamma_k^\wedge)$$

where  $\Gamma_k^\wedge$  consists of forms having degrees not exceeding  $k$ . These spaces form a filtration of  $\Omega(P)$ , compatible with the graded-differential  $*$ -structure, in the sense that

$$\Omega_k(P)^* = \Omega_k(P) \quad d\Omega_k(P) \subseteq \Omega_{k+1}(P) \quad \Omega_k(P) = \sum_{j \geq 0}^{\oplus} \Omega_k^j(P).$$

Let us consider a graded-differential  $*$ -algebra

$$\Pi(P) = \sum_{k \geq 0}^{\oplus} \Omega_k(P),$$

where the grading is given by numbers  $k$ , and the differential  $*$ -structure is induced from  $\Omega(P)$ . Let  $\mathfrak{r}: \Pi(P) \rightarrow \Pi(P)$  be the first-order map induced by the inclusions  $\Omega_k(P) \subseteq \Omega_{k+1}(P)$ . By definition, this is a monomorphism of differential  $*$ -algebras. Let  $\mathfrak{gr}(P)$  be the graded-differential  $*$ -algebra associated to the introduced filtration. In other words, we have a short exact sequence

$$(3.5) \quad 0 \rightarrow \Pi(P) \xrightarrow{\mathfrak{r}} \Pi(P) \rightarrow \mathfrak{gr}(P) \rightarrow 0$$

of differential  $*$ -algebras.

The space  $\Omega_k(P)$  is linearly spanned by elements of the form

$$w = \varphi \omega(\vartheta_1) \dots \omega(\vartheta_j)$$

where  $\varphi \in \mathfrak{hor}(P)$  and  $j \leq k$ , while  $\omega$  is an arbitrary connection. The algebra  $\mathfrak{gr}(P)$  is invariantly isomorphic to the algebra  $\mathfrak{vh}(P)$  of “vertically-horizontally” decomposed forms. Explicitly, the isomorphism  $\mathfrak{vh}(P) \leftrightarrow \mathfrak{gr}(P)$  is given by

$$(\varphi \otimes (\vartheta_1 \dots \vartheta_k)) \leftrightarrow [\varphi \omega(\vartheta_1) \dots \omega(\vartheta_k) + \Omega_{k-1}(P)].$$

In what follows it will be assumed that the two algebras are identified, with the help of the above isomorphism.

The factor-differential  $d_{\mathfrak{vh}}: \mathfrak{vh}(P) \rightarrow \mathfrak{vh}(P)$  is given by

$$d_{\mathfrak{vh}}(\varphi \otimes \vartheta) = (-1)^{\partial \varphi} \sum_k \varphi_k \otimes \pi(c_k) \vartheta + (-1)^{\partial \varphi} \varphi \otimes d(\vartheta)$$

where  $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$ . The following technical lemma will prove useful in the computation of the cohomology algebra of  $\mathfrak{vh}(P)$ .



**Lemma 3.5.** *For a given tensorial form  $\lambda: \Gamma_{inv} \rightarrow \Omega(P)$  let us define linear maps  $S_\lambda, T_\lambda: \mathfrak{vh}(P) \rightarrow \mathfrak{vh}(P)$  by the following equalities*

$$S_\lambda(\varphi \otimes (\vartheta_1 \dots \vartheta_n)) = \sum_k \varphi_k \lambda \pi(c_k) \otimes (\vartheta_1 \dots \vartheta_n)$$

$$T_\lambda(\varphi \otimes (\vartheta_1 \dots \vartheta_n)) = (-1)^{\partial \varphi} \varphi \sum_{i=1}^n (-1)^{i-1} \lambda(\vartheta_{il}) \otimes \left\{ \vartheta_1 \dots \vartheta_{i-1} \right\} \circ c_{il} \left\{ \vartheta_{i+1} \dots \vartheta_n \right\}$$

where  $\vartheta_i \in \Gamma_{inv}$  and  $\varpi(\vartheta_i) = \sum_l \vartheta_{il} \otimes c_{il}$ . We have then

$$(3.6) \quad S_\lambda = T_\lambda d_{th} + (-1)^{\partial \lambda} d_{th} T_\lambda.$$

*Proof.* A direct computation gives

$$\begin{aligned} T_\lambda d_{th}(\varphi \otimes \vartheta) &= (-1)^{\partial \varphi} \sum_k T_\lambda(\varphi_k \otimes \pi(c_k) \vartheta) + (-1)^{\partial \varphi} T_\lambda(\varphi \otimes d(\vartheta)) \\ &= \sum_k \varphi_k \lambda \pi(c_k) \otimes \vartheta + \sum_{ikl} (-1)^i \varphi_k \lambda(\vartheta_{il}) \otimes \left\{ \pi(c_k) \vartheta_1 \dots \vartheta_{i-1} \right\} \circ c_{il} \left\{ \vartheta_{i+1} \dots \vartheta_n \right\} \\ &\quad + \sum_{i < j} \sum_l (-1)^{i+j} \varphi \lambda(\vartheta_{il}) \otimes \left\{ \vartheta_1 \dots \vartheta_{i-1} \right\} \circ c_{il} \left\{ \vartheta_{i+1} \dots d(\vartheta_j) \dots \vartheta_n \right\} \\ &\quad + \sum_{i > j} \sum_l (-1)^{i+j-1} \varphi \lambda(\vartheta_{il}) \otimes \left\{ \vartheta_1 \dots d(\vartheta_j) \dots \vartheta_{i-1} \right\} \circ c_{il} \left\{ \vartheta_{i+1} \dots \vartheta_n \right\} \\ &\quad - \sum_{il} \varphi \lambda(\vartheta_{il}) \otimes \left\{ \vartheta_1 \dots \vartheta_{i-1} \right\} \circ c_{il}^{(1)} \left\{ \pi(c_{il}^{(2)}) \vartheta_{i+1} \dots \vartheta_n \right\}, \end{aligned}$$

where  $\vartheta = \vartheta_1 \dots \vartheta_n$ . On the other hand

$$\begin{aligned} d_{th} T_\lambda(\varphi \otimes \vartheta) &= (-1)^{\partial \lambda + i - 1} \varphi \sum_{il} \lambda(\vartheta_{il}) \otimes d \left[ \left\{ \vartheta_1 \dots \vartheta_{i-1} \right\} \circ c_{il} \left\{ \vartheta_{i+1} \dots \vartheta_n \right\} \right] \\ &\quad + (-1)^{\partial \lambda + i - 1} \sum_{ikl} \varphi_k \lambda(\vartheta_{il}) \otimes \left[ \pi(c_k c_{il}^{(1)}) \left\{ \vartheta_1 \dots \vartheta_{i-1} \right\} \circ c_{il}^{(2)} \left\{ \vartheta_{i+1} \dots \vartheta_n \right\} \right]. \end{aligned}$$

Combining above expressions and performing elementary further transformations we conclude that (3.6) holds.  $\square$

According to the above lemma, the map  $S_\lambda$  is cohomologically trivial (it maps  $d_{th}$ -cocycles into coboundaries).

The algebra  $\mathfrak{vh}(P)$  can be naturally decomposed in the following way

$$(3.7) \quad \mathfrak{vh}(P) = \sum_{u \in \mathcal{T}}^{\oplus} \mathcal{F}_u \otimes \Lambda_u,$$

where  $\Lambda_u = H_u \otimes \Gamma_{inv}^\wedge$  are (understandable as) subcomplexes, with the differentials specified by

$$d_u(e_i \otimes \eta) = e_i \otimes d(\eta) + \sum_{j=1}^n e_j \otimes \pi(u_{ji}) \eta.$$

**Lemma 3.6.** *If  $u \in \mathcal{T}$  is such that  $\pi(u) \neq 0$  then the complex  $\Lambda_u$  is acyclic.*

*Proof.* The maps  $S_\lambda$  trivially act on factors  $\Lambda_u$ , figuring in the decomposition (3.7), and they act via composition on spaces  $\mathcal{F}_u$ . Therefore, according to Lemma 3.5, the cohomology of the complex  $\Lambda_u$  will be trivial whenever there exists  $\lambda$  satisfying  $S_\lambda(\mathcal{F}_u) \neq \{0\}$ .  $\square$

Let  $\mathfrak{h}_M$  be a space consisting of horizontal forms  $\varphi$  satisfying

$$(\text{id} \otimes \pi)F^\wedge(\varphi) = 0.$$

In other words,

$$\mathfrak{h}_M = \sum_u^* \mathcal{H}^u(P)$$

where the summation is performed over  $u \in \mathcal{T}$  satisfying  $\pi(u) = \{0\}$ . Evidently,  $\Omega(M) \subseteq \mathfrak{h}_M$ .

**Lemma 3.7.** *The space  $\mathfrak{h}_M$  is a graded  $*$ -subalgebra of  $\mathfrak{hor}(P)$ , closed under the action of the differential map.*  $\square$

The elements of  $\mathfrak{h}_M$  play the role of horizontal forms invariant under infinitesimal vertical diffeomorphisms.

Let  $H_{\mathfrak{th}}(P)$  be the cohomology algebra of  $\mathfrak{th}(P)$ . Having in mind that if  $\pi(u) = \{0\}$  then the differential  $d_u$  reduces to the multiple of the ordinary differential in  $\Gamma_{\text{inv}}^\wedge$  it follows that

$$(3.8) \quad H_{\mathfrak{th}}(P) = \mathfrak{h}_M \otimes H(\Gamma_{\text{inv}}^\wedge).$$

Let  $E(P) = \{E_r(P) | r \in \mathbb{N}\}$  be the spectral sequence associated to the short exact sequence (3.5). The introduced filtration of  $\Omega(P)$  induces a filtration of the  $*$ -algebra  $H(P)$  of cohomology classes. We have

$$H_k(P) = \sum_j^\oplus H_k^j(P).$$

Applying general theory [3], it follows that the introduced spectral sequence is convergent, and that  $E_\infty(P)$  coincides with the graded  $*$ -algebra associated to the filtered  $H(P)$ .

By construction, the first cohomology algebra is given by  $E_1(P) = H_{\mathfrak{th}}(P)$ .

**Lemma 3.8.** *The differential  $d_1$  is given by*

$$(3.9) \quad d_1(w \otimes \vartheta) = dw \otimes \vartheta.$$

*Proof.* Let us fix a connection  $\omega$ , and let  $\psi = \sum_{\varphi\vartheta} \varphi \otimes [\vartheta]^\wedge$  be a  $d_{\mathfrak{th}}$ -cocycle. Here  $\varphi \in \mathfrak{h}_M$  and  $\vartheta = \vartheta_1 \otimes \dots \otimes \vartheta_n$ , with  $\vartheta_i \in \Gamma_{\text{inv}}$ .

Then both forms  $w = \sum_{\varphi\vartheta} \varphi \omega(\vartheta_1) \dots \omega(\vartheta_n)$  and  $dw$  belong to  $\Omega_n(P)$ . By definition,  $d_1[\psi]$  will be represented by the image of  $dw$  in  $\mathfrak{th}^n(P)$ . We have

$$\begin{aligned} dw &= \sum_{\varphi\vartheta} (-1)^{\partial\varphi} \varphi \sum_{i=1}^n (-1)^{i-1} \omega(\vartheta_1) \dots \omega(\vartheta_{i-1}) \langle \omega, \omega \rangle (\vartheta_i) \omega(\vartheta_{i+1}) \dots \omega(\vartheta_n) \\ &+ \sum_{\varphi\vartheta} d\varphi \omega^\otimes(\vartheta) + (-1)^{\partial\varphi} \varphi \sum_{i=1}^n (-1)^{i-1} \omega(\vartheta_1) \dots \omega(\vartheta_{i-1}) R_\omega(\vartheta_i) \omega(\vartheta_{i+1}) \dots \omega(\vartheta_n), \end{aligned}$$

and it follows that  $dw - \sum_{\varphi\vartheta} d\varphi\omega(\vartheta_1) \dots \omega(\vartheta_n)$  belongs to  $\Omega_{n-1}(P)$ . Hence,

$$d_1[\psi] = \sum_{\varphi\vartheta} d\varphi \otimes [\vartheta]$$

which completes the proof.  $\square$

In particular,

$$(3.10) \quad E_2(P) = H(\mathfrak{h}_M) \otimes H(\Gamma_{inv}^\wedge).$$

In a special case when  $G$  is “connected,” only scalar elements of  $\mathcal{A}$  are annihilated by the differential map, and we have

$$\Omega(M) = \mathfrak{h}_M \quad H(M) = H(\mathfrak{h}_M).$$

Such a connectedness assumption is equivalent to the statement that only trivial representation  $\emptyset$  is annihilated by  $\pi$ .

### 3.4. The Universal Construction

In this subsection a general approach to quantum characteristic classes will be presented. It is based on a cohomological interpretation of the domain of the Weil homomorphism. We shall consider two levels of generality—quantum principal bundles with general connections, and structures admitting regular connections. As first, a characterization the image of the Weil homomorphism will be given.

**Proposition 3.9.** *Let us assume that  $\omega$  is a regular and multiplicative connection on  $P$ . Then the image of the map  $W^\omega$  consists of differential forms on  $M$  which are expressible in terms of  $\omega$  and  $d\omega$ .*

*Proof.* If  $\omega$  is regular and multiplicative then horizontal elements expressible via  $\omega$  and  $d\omega$  are of the form  $w = R_\omega^\otimes(\vartheta)$ , where  $\vartheta \in \Gamma_{inv}^\otimes$  is arbitrary. This follows from the multiplicativity of the horizontal projection  $h_\omega$ , and the definition of the curvature map. If  $w \in \Omega(M)$  then it is possible to assume that  $\vartheta \in I^*$ , because of the tensoriality of  $R_\omega$ .  $\square$

In generalizing the formalism to the level of arbitrary bundles, we shall follow *the idea of universality*. Algebraic expressions generating characteristic classes should be the same for all bundles.

In what follows it will be assumed that the higher-order calculus on  $G$  is described by one of the two “extreme” algebras—by the universal envelope  $\Gamma^\wedge$  corresponding to the maximal solution, or the bicovariant [14] exterior algebra  $\Gamma^\vee$ , which describes the minimal appropriate higher-order calculus. Both cases can be treated essentially in the same way, and we shall commonly pass from one algebra to another. We shall denote by  $d^\wedge$  and  $d^\vee$  the corresponding differentials.

By definition

$$\Gamma^{\vee n} = \Gamma^{\otimes n} / \ker(A_n),$$

where  $A_n : \Gamma^{\otimes n} \rightarrow \Gamma^{\otimes n}$  are corresponding braided antisymmetrizers, given by

$$A_n = \sum_{\pi \in S_n} (-1)^\pi \sigma_\pi$$

where operators  $\sigma_\pi$  are constructed by replacing transpositions figuring in minimal decompositions of  $\pi$ , by the corresponding  $\sigma$ -twists (and  $\sigma$  is acting on  $\Gamma^{\otimes 2}$  as a bimodule automorphism). Furthermore,

$$A_{k+l} = (A_k \otimes A_l)A_{kl} \quad A_{kl} = \sum_{\pi \in S_{kl}} (-1)^\pi \sigma_{\pi^{-1}},$$

where  $S_{kl} \subseteq S_{k+l}$  consists of permutations preserving the order of the first  $k$  and the last  $l$  factors. It is interesting to observe that maps  $A_{kl}$  preserve the ideal  $S^\wedge$ , and hence there exist natural factorizations  $A_{kl}^\wedge: \Gamma^\wedge \rightarrow \Gamma^\wedge$ .

The coproduct map is extendible to a homomorphism  $\phi^\vee: \Gamma^\vee \rightarrow \Gamma^\vee \widehat{\otimes} \Gamma^\vee$  of graded-differential  $*$ -algebras. By universality, there exists the unique graded-differential homomorphism  $\check{\gamma}: \Gamma^\wedge \rightarrow \Gamma^\vee$  reducing to identity maps on  $\mathcal{A}$  and  $\Gamma$ . We have

$$\phi^\vee \check{\gamma} = (\check{\gamma} \otimes \check{\gamma}) \widehat{\phi}.$$

Moreover, all constructions of this paper can be performed also for “intermediate” higher-order calculi, described by arbitrary higher-order graded-differential  $*$ -ideals  $S_\star \subseteq \Gamma^\wedge$  satisfying

$$\widehat{\phi}(S_\star) \subseteq S_\star \widehat{\otimes} \Gamma^\wedge + \Gamma^\wedge \widehat{\otimes} S_\star.$$

The above condition implies  $S_\star \subseteq [\ker(A)]^\wedge$ , which explains the minimality of the braided exterior algebra.

It is important to mention that switching from one higher-order calculus over  $G$  to another may influence drastically the properties of the corresponding characteristic classes for  $M$ . We shall return to this point in Section 8.

Let  $\Omega$  be the universal differential envelope of the tensor algebra  $\Gamma_{inv}^\otimes$ . The algebra  $\Omega$  will be endowed with a grading specified by requiring that elements from  $\Gamma_{inv}$  have degree 1. By definition,  $\Omega$  is generated (as a differential algebra) by the space  $\Gamma_{inv}$  with the only relation  $d(1) = 0$ . The  $*$ -involution on  $\Gamma_{inv}$  naturally extends to  $\Omega$ , so that  $d: \Omega \rightarrow \Omega$  is a hermitian map. We have

$$(3.11) \quad H(\Omega) = \mathbb{C},$$

which is a property of all universal differential envelopes (of algebras).

The universality property of  $\Omega$  implies that there exists the unique graded-differential homomorphism  $\widetilde{\omega}: \Omega \rightarrow \Omega \widehat{\otimes} \Gamma^\wedge$  satisfying

$$(3.12) \quad \widetilde{\omega}(\vartheta) = \omega(\vartheta) + 1 \otimes \vartheta$$

for each  $\vartheta \in \Gamma_{inv}$ . This map is hermitian, and satisfies the following equality

$$(3.13) \quad (\widetilde{\omega} \otimes \text{id})\widetilde{\omega} = (\text{id} \otimes \widehat{\phi})\widetilde{\omega}.$$

It is also possible to introduce a natural right action  $\varpi_\wedge: \Omega \rightarrow \Omega \otimes \mathcal{A}$ , extending the adjoint action map  $\varpi$ . We have

$$\varpi_\wedge = (\text{id} \otimes p_*)\widetilde{\omega}.$$

Let  $\mathcal{T} \subseteq \Omega$  be a graded-differential  $*$ -subalgebra consisting of  $\widetilde{\omega}$ -invariant elements. In other words  $w \in \mathcal{T}$  iff  $\widetilde{\omega}(w) = w \otimes 1$ .

Let  $P = (\mathcal{B}, i, F)$  be a quantum principal  $G$ -bundle over  $M$ , endowed with a calculus  $\Omega(P)$ . Let  $\omega$  be an arbitrary connection on  $P$ .

**Proposition 3.10.** (i) *There exists the unique homomorphism  $\widehat{\omega}: \Omega \rightarrow \Omega(P)$  of differential algebras satisfying  $\widehat{\omega}(\vartheta) = \omega(\vartheta)$  for each  $\vartheta \in \Gamma_{inv}$ . This map is hermitian, and the diagram*

$$(3.14) \quad \begin{array}{ccc} \Omega & \xrightarrow{\widetilde{\omega}} & \Omega \widehat{\otimes} \Gamma^\wedge \\ \widehat{\omega} \downarrow & & \downarrow \widehat{\omega} \otimes \text{id} \\ \Omega(P) & \xrightarrow{\widehat{F}} & \Omega(P) \widehat{\otimes} \Gamma^\wedge \end{array}$$

*is commutative. In particular, it follows that  $\widehat{\omega}(\mathfrak{I}) \subseteq \Omega(M)$ .*

(ii) *The induced cohomology map  $W: H(\mathfrak{I}) \rightarrow H(M)$  is independent of the choice of  $\omega$ .*

*Proof.* Property (i) follows from the definition of  $\widetilde{\omega}$ , equation (2.1), and the universality of  $\Omega$ . Let  $\tau$  be another connection on  $P$ , and let  $L: \Omega \rightarrow \Omega(P)$  be a linear map specified by

$$(3.15) \quad \begin{aligned} L(\vartheta) &= 0 & L(d\vartheta) &= \varphi(\vartheta) & L(1) &= 0 \\ L(wu) &= L(w)\widehat{\tau}(u) + (-1)^{\partial w}\widehat{\omega}(w)L(u), \end{aligned}$$

where  $\varphi = \tau - \omega$ . A direct computation gives

$$(3.16) \quad \widehat{\omega} - \widehat{\tau} = Ld + dL,$$

and moreover the diagram

$$(3.17) \quad \begin{array}{ccc} \Omega & \xrightarrow{L} & \Omega(P) \\ \widetilde{\omega} \downarrow & & \downarrow \widehat{F} \\ \Omega \widehat{\otimes} \Gamma^\wedge & \xrightarrow{L \otimes \text{id}} & \Omega(P) \widehat{\otimes} \Gamma^\wedge \end{array}$$

is commutative. Hence,  $L(\mathfrak{I}) \subseteq \Omega(M)$ . In particular, it follows that  $\widehat{\omega}$  and  $\widehat{\tau}$  induce the same cohomology map  $W: H(\mathfrak{I}) \rightarrow H(M)$ .  $\square$

The constructed map  $W$  is a counterpart of Weil homomorphism, at the level of general quantum principal bundles. Characteristic classes are therefore labeled by the elements of  $H(\mathfrak{I})$ .

The algebra  $\Omega$  possesses various properties characteristic to differential algebras describing the calculus on quantum principal bundles (however, here there will be no analogs of the base space and the bundle, because  $\Omega^0 = \mathbb{C}$ ). In particular, it is possible to introduce a natural decomposition

$$(3.18) \quad \Omega \leftrightarrow \mathfrak{h}(\Omega) \otimes \Gamma_{inv}^\wedge = \mathfrak{v}\mathfrak{h}(\Omega) \quad \varphi\iota(\vartheta) \leftrightarrow \varphi \otimes \vartheta$$

where  $\mathfrak{h}(\Omega) \subseteq \Omega$  is a graded  $*$ -subalgebra describing “horizontal elements,” defined by

$$\mathfrak{h}(\Omega) = \widetilde{\omega}^{-1}(\Omega \otimes \mathcal{A}).$$

It follows that  $\varpi_\wedge[\mathfrak{h}(\Omega)] \subseteq \mathfrak{h}(\Omega) \otimes \mathcal{A}$ , in other words  $\mathfrak{h}(\Omega)$  is  $\varpi_\wedge$ -invariant. The algebra  $\mathfrak{I}$  is the  $\varpi_\wedge$ -fixed point subalgebra of  $\mathfrak{h}(\Omega)$ .

Similarly,  $\mathfrak{vh}(\Omega)$  possesses a natural differential  $*$ -algebra structure, in accordance with the analogy with vertically-horizontally decomposed forms.

Following the ideas of the previous subsection, it is possible to associate a natural spectral sequence  $E(\Omega) = \{E_r(\Omega) | r \in \mathbb{N}\}$  to the algebra  $\Omega$ , so that

$$E_1(\Omega) = H(\mathfrak{vh}(\Omega), d_{\mathfrak{vh}}) = \mathbb{T} \otimes H(G) \quad E_2(\Omega) = H(\mathbb{T}) \otimes H(G),$$

and the sequence converges to the trivial cohomology algebra  $E_\infty(\Omega) = H(\Omega) = \mathbb{C}$ . In particular, if  $H(G) = \mathbb{C}$  then  $\mathbb{T}^+$  will be acyclic, too. In this particular case there will be no intrinsic characteristic classes (the level of general bundles and differential structures on them).

The following is a prescription of constructing the cocycles from  $\mathbb{T}$ . Every cocycle  $w \in \mathbb{T}^+$  is of the form  $w = d\varphi$ , where  $\varphi$  is some  $\varpi_\wedge$ -invariant element of  $\Omega$ . Then we have the equivalence

$$w \in \mathbb{T} \iff d\{\tilde{\omega}(\varphi) - \varphi \otimes 1\} = 0.$$

For every quantum principal bundle  $P$ , the cocycles representing characteristic classes are exact, as classes on the bundle, with  $\hat{\omega}(w) = d\hat{\omega}(\varphi)$ . Therefore, the above introduced elements  $\varphi$  play the role of *universal* Chern-Simons forms. At the level of cohomology classes,

**Lemma 3.11.** *Let  $\mathcal{C} \subseteq \Omega \otimes \Gamma^{\wedge+}$  be the a subcomplex spanned by elements of the form  $c = \tilde{\omega}(\varphi) - \varphi \otimes 1$ , where  $\varphi \in \Omega$  is  $\varpi_\wedge$ -invariant. Then the following natural correspondence holds*

$$(3.19) \quad H^n(\mathbb{T}) \leftrightarrow H^{n-1}(\mathcal{C}).$$

*Proof.* Let us consider the following natural short exact sequence of complexes

$$0 \rightarrow \mathbb{T} \rightarrow I(\Omega) \rightarrow \mathcal{C} \rightarrow 0$$

where  $I(\Omega) \subseteq \Omega$  is the subalgebra of  $\varpi_\wedge$ -invariant elements. Passing to the corresponding long exact sequence, and observing that  $H[I(\Omega)] = \mathbb{C}$  we conclude that (3.19) holds.  $\square$

Let us compute explicitly *the cocycles* of  $\mathbb{T}$  in dimensions 2, 3 and 4. Clearly, closed elements of  $\mathbb{T}^2$  are of the form  $w = d\vartheta$ , where  $\vartheta \in \Gamma_{inv}$  is  $\varpi$ -invariant and closed in  $\Gamma_{inv}^\wedge$ . It is interesting to observe that if the higher-order calculus on  $G$  is described by the braided exterior algebra  $\Gamma^\vee$ , the elements of  $H^2(\mathbb{T})$  are labeled by all  $\varpi$ -invariant elements of  $\Gamma_{inv}$ , since such elements are automatically closed.

Furthermore, closed elements of  $\mathbb{T}^3$  are generated by Chern-Simons forms

$$\varphi = \sum_{\vartheta\eta} \vartheta \otimes \eta,$$

where  $\vartheta, \eta \in \Gamma_{inv}$ , such that

$$d^\wedge[\varphi]^\wedge = 0 \quad \sigma(\varphi) = \varphi.$$

In general, the corresponding cocycles will be non-trivial in  $\mathbb{T}$ , which is an interesting purely quantum phenomena. There exist characteristic classes in *odd* dimensions. However, if the higher-order calculus on  $G$  is described by  $\Gamma^\vee$  then  $H^3(\mathbb{T}) = \{0\}$ .

Third-order Chern-Simons forms can be written as

$$\varphi = \psi + \sum_{\vartheta\eta} R(\vartheta)\eta,$$

with  $\varpi$ -invariant components  $\sum_{\vartheta\eta} \vartheta \otimes \eta \in \Gamma_{inv}^{\otimes 2}$  and  $\psi \in \Gamma_{inv}^{\otimes 3}$ . In the above formula

$$(3.20) \quad R(\vartheta) = d\vartheta - \delta(\vartheta)$$

is the *universal curvature form*. A straightforward calculation shows that requiring the horizontality of  $w = d\varphi$  is equivalent to the following system of equations

$$\begin{aligned} \sum \delta(\vartheta) \otimes \eta &= \sum xy \otimes z & d^\wedge[\psi]^\wedge &= 0 \\ \sum \{c^\top(\vartheta)\eta + \vartheta \otimes d^\wedge(\eta)\} + \sum p \otimes qr &= 0 \\ \sum c^\top(p)qr + p \otimes d^\wedge(qr) &= 0 \\ \sum \{c^\top(xy)z + xy \otimes d^\wedge(z)\} + \sum \delta(p) \otimes qr &= 0, \end{aligned}$$

where  $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$  and

$$\sum x \otimes y \otimes z = A_{21}(\psi) \quad \sum p \otimes q \otimes r = A_{12}(\psi).$$

Now we give the complete description of the horizontal algebra  $\mathfrak{h}(\Omega)$ . The algebra  $\mathfrak{h}(\Omega)$  is closed under the operation

$$(3.21) \quad \ell_\star(\vartheta, \varphi) = \vartheta\varphi - (-1)^{\vartheta\varphi} \sum_k \varphi_k(\vartheta \circ c_k),$$

where  $\vartheta \in \Gamma_{inv}$  and  $\sum_k \varphi_k \otimes c_k = \varpi_\wedge(\varphi)$ .

**Proposition 3.12.** (i) *Let us assume that the higher-order calculus on  $G$  is described by the universal envelope  $\Gamma^\wedge$ . Then  $\mathfrak{h}(\Omega)$  is the minimal  $\ell_\star$ -invariant subalgebra of  $\Omega$  containing  $R(\vartheta)$  and the elements from  $S_{inv}^{\wedge 2}$ .*

(ii) *If  $\Gamma^\vee$  describes the higher-order calculus on  $G$  then  $\mathfrak{h}(\Omega)$  is linearly generated by “elementary” horizontal forms  $w \in \Gamma_{inv}^\otimes$  characterized by*

$$(3.22) \quad A_{*1}(w) = 0,$$

*together with the elements obtained from  $R(\vartheta)$  by the multiple actions of  $\ell_\star$ .*

*Proof.* Essentially the same reasoning as in [6]-Section 3, where the structure of horizontal forms was analyzed in details.  $\square$

Let  $\sqcup: \Omega \rightarrow \Omega \otimes \Gamma_{inv}$  be the “vertical contraction” map, obtained by naturally projecting the values of  $\varpi$  on  $\Omega \otimes \Gamma_{inv}$ . Then we have

$$(3.23) \quad \mathfrak{h}(\Omega) = \ker(\sqcup),$$

under the assumption that the whole calculus on  $G$  is described by  $\Gamma^\vee$ . This means that the system of equations describing Chern-Simons forms reduces to the first-order equation (on the second degree).

In any case, these equations always appear as algebraic expressions in spaces  $\Gamma_{inv}^\otimes \otimes \Gamma_{inv}^{\wedge, \vee}$ , involving maps  $\sigma, \delta, \varpi, c^\top$  and the differential  $d^{\wedge \vee}$  in the second factor.

Let us return to the regular case. Essentially the same cohomological description of the domain of the Weil homomorphism is possible for bundles admitting regular (and multiplicative) connections. The only difference is that the algebra  $\Omega$  should be factorized through the appropriate ideal, which takes into account the characteristic property of regular (and multiplicative) connections.

Let  $\mathcal{K} \subseteq \Omega$  be the ideal generated by elements from  $S_{inv}^{\wedge 2}$ , and the elements of the form

$$(3.24) \quad j_3(\eta, \vartheta) = \ell_\star(\eta, R(\vartheta)) = \eta R(\vartheta) - \sum_k R(\vartheta_k)(\eta \circ c_k)$$

$$(3.25) \quad j_4(\eta, \vartheta) = R(\eta)R(\vartheta) - \sum_k R(\vartheta_k)R(\eta \circ c_k),$$

where  $\eta, \vartheta \in \Gamma_{inv}$  and  $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$ .

It follows that  $\Gamma_{inv}^\wedge$  is a subalgebra of  $\Omega_\star = \Omega/\mathcal{K}$ , in a natural manner. We have

$$(3.26) \quad \widetilde{\varpi}R(\vartheta) = \sum_k R(\vartheta_k) \otimes c_k$$

$$(3.27) \quad R(\vartheta)^* = R(\vartheta^*).$$

It is easy to see that the ideal  $\mathcal{K}$  is  $\widetilde{\varpi}, *$ -invariant. Moreover,

**Lemma 3.13.** *The ideal  $\mathcal{K}$  is  $d$ -invariant.*

*Proof.* We have to check that the relations defining  $\mathcal{K}$  are compatible with the action of  $d$ . Let us first observe that

$$(3.28) \quad dR(\vartheta) = \sum_k R(\vartheta_k)\pi(c_k) + w$$

where  $w \in \mathcal{K}$ . Also, the first argument in  $j_3$  can be extended to the whole  $\Gamma_{inv}^\wedge$ . A direct computation gives

$$\begin{aligned} dj_3(\eta, \vartheta) &= w_1 + d\eta R(\vartheta) - \sum_k R(\vartheta_k)d(\eta \circ c_k) - \sum_k \eta R(\vartheta_k)\pi(c_k) \\ &\quad - \sum_k R(\vartheta_k)\pi(c_k^{(1)})(\eta \circ c_k^{(2)}) \\ &= w_1 + j_4(\eta, \vartheta) + d^\wedge \eta R(\vartheta) - \sum_k R(\vartheta_k)d^\wedge(\eta \circ c_k) - \sum_k \eta R(\vartheta_k)\pi(c_k) \\ &\quad - \sum_k R(\vartheta_k)\pi(c_k^{(1)})(\eta \circ c_k^{(2)}) \\ &= w_1 + j_4(\eta, \vartheta) + j_3(d^\wedge \eta, \vartheta) + \sum_k \left( R(\vartheta_k)(\eta \circ c_k^{(1)})\pi(c_k^{(2)}) - \eta R(\vartheta_k)\pi(c_k) \right) \\ &= w_1 + j_4(\eta, \vartheta) + j_3(d^\wedge \eta, \vartheta) - \sum_k j_3(\eta, \vartheta_k)\pi(c_k). \end{aligned}$$



Applying similar transformations we obtain

$$\begin{aligned}
dj_4(\eta, \vartheta) &= w_2 + \sum_l R(\eta_l) \pi(d_l) R(\vartheta) - \sum_k R(\vartheta_k) \pi(c_k^{(1)}) R(\eta \circ c_k^{(2)}) \\
&\quad + R(\eta) \sum_k R(\vartheta_k) \pi(c_k) - \sum_{kl} R(\vartheta_k) R(\eta_l \circ c_k^{(2)}) \pi[\kappa(c_k^{(1)}) d_l c_k^{(3)}] \\
&= w_3 + \sum_{kl} j_4(\eta_l, \vartheta_k) \pi(d_l c_k),
\end{aligned}$$

where  $w_i \in \mathcal{K}$  and  $\varpi(\eta) = \sum_l \eta_l \otimes d_l$ . Finally,

$$d(\pi(a^{(1)}) \otimes \pi(a^{(2)})) = -j_3(\pi(a^{(1)}), \pi(a^{(2)}))$$

for each  $a \in \mathcal{R}$ , in other words  $d(S_{inv}^\wedge) \subseteq \mathcal{K}$ .  $\square$

The maps  $\tilde{\omega}, d, *, \varpi_\wedge$  are hence projectable to  $\Omega_*$ . Obviously, projected maps (that will be denoted by the same symbols) are in the same algebraic relations as the original ones. Let us introduce the horizontal part  $\mathfrak{h}(\Omega_*) = \tilde{\omega}^{-1}(\Omega_* \otimes \mathcal{A})$  of  $\Omega_*$ , which is a  $\varpi_\wedge$ -invariant  $*$ -subalgebra of  $\Omega_*$ . Applying a similar reasoning as in [6] it follows that

**Lemma 3.14.** (i) *The product map in  $\Omega_*$  induces a graded vector space isomorphism*

$$(3.29) \quad \mathfrak{h}(\Omega_*) \otimes \Gamma_{inv}^\wedge \leftrightarrow \Omega_*.$$

(ii) *The map  $R: \Gamma_{inv} \rightarrow \Omega_*$  can be uniquely extended to a unital  $*$ -homomorphism  $R: \Sigma \rightarrow \Omega_*$ . The extended  $R$  maps isomorphically  $\Sigma$  onto  $\mathfrak{h}(\Omega_*)$ . Moreover, the diagram*

$$(3.30) \quad \begin{array}{ccc} \Sigma & \xrightarrow{R} & \mathfrak{h}(\Omega_*) \\ \varpi_\Sigma \downarrow & & \downarrow \varpi_\wedge \\ \Sigma \otimes \mathcal{A} & \xrightarrow{R \otimes \text{id}} & \mathfrak{h}(\Omega_*) \otimes \mathcal{A} \end{array}$$

*is commutative.*  $\square$

Inductively applying (3.28) and relations given by  $j_3$ , it follows that

$$d\varphi = \sum_k \varphi_k \pi(c_k)$$

for each  $\varphi \in \mathfrak{h}(\Omega_*)$ , where  $\varpi_\wedge(\varphi) = \sum_k \varphi_k \otimes c_k$ . Let  $\mathfrak{J} \subseteq \Omega_*$  be the  $\tilde{\omega}$ -fixed-point subalgebra (equivalently, the  $\varpi_\wedge$ -fixed-point subalgebra of  $\mathfrak{h}(\Omega_*)$ ). From the above formula it follows that

$$d(\mathfrak{J}) = \{0\},$$

and hence

$$H(\mathfrak{J}) = \mathfrak{J} = I(\Sigma)$$

in a natural manner. This gives a connection with the previous definition of the Weil homomorphism. Essentially the same reasoning as in the proof of Proposition 3.10 leads to the following result.

**Proposition 3.15.** *Let us consider a quantum principal  $G$ -bundle  $P = (\mathcal{B}, i, F)$  over  $M$ , with a calculus  $\Omega(P)$  admitting regular and multiplicative connections. Let  $\omega: \Gamma_{inv} \rightarrow \Omega(P)$  be an arbitrary connection of this type.*

(i) *There exists the unique homomorphism  $\widehat{\omega}: \Omega_* \rightarrow \Omega(P)$  of differential algebras extending the identity map on  $\Gamma_{inv}$ . The map  $\widehat{\omega}$  is hermitian, and the diagram*

$$(3.31) \quad \begin{array}{ccc} \Omega_* & \xrightarrow{\widetilde{\omega}} & \Omega_* \widehat{\otimes} \Gamma^\wedge \\ \widehat{\omega} \downarrow & & \downarrow \widehat{\omega} \otimes \text{id} \\ \Omega(P) & \xrightarrow[\widehat{F}]{} & \Omega(P) \widehat{\otimes} \Gamma^\wedge \end{array}$$

*is commutative. In particular,  $\widehat{\omega}(\square) \subseteq \Omega(M)$ .*

(ii) *The following identities hold*

$$(3.32) \quad \widehat{\omega}R = R_\omega$$

$$(3.33) \quad \widehat{\omega}D = D_\omega \widehat{\omega}$$

*where  $D: \Omega_* \rightarrow \Omega_*$  is a first-order antiderivation specified by*

$$DR(\vartheta) = 0 \quad D\vartheta = R(\vartheta).$$

(iii) *The induced cohomology map  $W: H(\square) \rightarrow H(M)$  is independent of the choice of  $\omega$ .  $\square$*

The above introduced map  $D$  is the *universal covariant derivative*, in accordance with (3.33). We have  $D^2 = 0$ .

Let  $d_{\text{th}}: \Omega_* \rightarrow \Omega_*$  be “the universal” vertical differential. By definition, this map is acting in the following way

$$(3.34) \quad d_{\text{th}}(\varphi \otimes \vartheta) = \sum_k \varphi_k \otimes \pi(c_k)\vartheta + \varphi \otimes d^\wedge(\vartheta),$$

where  $\varpi_\wedge(\varphi) = \sum_k \varphi_k \otimes c_k$ .

**Lemma 3.16.** *The following identities hold*

$$(3.35) \quad Dd_{\text{th}} + d_{\text{th}}D = 0$$

$$(3.36) \quad D + d_{\text{th}} = d.$$

*Proof.* It is sufficient to check the first identity on elements of the form  $\vartheta, R(\vartheta)$  where  $\vartheta \in \Gamma_{inv}$ . We have

$$\begin{aligned} d_{\text{th}}D(\vartheta) &= \sum_k R(\vartheta_k) \otimes \pi(c_k) = (R \otimes \text{id})\sigma\delta(\vartheta) - (R \otimes \text{id})\delta(\vartheta) \\ &= (\text{id} \otimes R)\delta(\vartheta) - (R \otimes \text{id})\delta(\vartheta) = -Dd_{\text{th}}(\vartheta). \end{aligned}$$

Furthermore

$$Dd_{\mathfrak{th}}R(\vartheta) = \sum_k D(R(\vartheta_k) \otimes \pi(c_k)) = \sum_k R(\vartheta_k)R\pi(c_k) = 0 = -d_{\mathfrak{th}}DR(\vartheta).$$

Finally, the second equality restricted to  $\Gamma_{inv}$  gives us the definition of  $R$ .  $\square$

Let us observe that the map  $D$  acts skew-diagonally, with respect to a natural bigrading in  $\Omega_*$ . This implies that

$$H_D(\Omega_*) = \sum_{k,l \in \mathbb{Z}}^{\oplus} H^{kl}(\Omega_*)$$

with the induced bigrading.

To conclude this section, let us analyze the relation between cohomology algebras  $H(\mathfrak{I})$  and  $H(\mathfrak{J}) = \mathfrak{J}$ . Let  $p_{\mathcal{K}}: \Omega \rightarrow \Omega_* = \Omega/\mathcal{K}$  be the factor projection map. This map intertwines the corresponding  $\tilde{\omega}$ . In particular,  $p_{\mathcal{K}}(\mathfrak{I}) \subseteq \mathfrak{J}$ .

In the general case two cohomology algebras will be related through the long exact sequence

$$(3.37) \quad \rightarrow H(\mathcal{K} \cap \mathfrak{I}) \rightarrow H(\mathfrak{I}) \rightarrow H(\mathfrak{J}) \rightarrow H(\mathcal{K} \cap \mathfrak{I}) \rightarrow$$

associated to

$$0 \rightarrow \mathcal{K} \cap \mathfrak{I} \rightarrow \mathfrak{I} \rightarrow \mathfrak{J} \rightarrow 0.$$

The ideal  $\mathcal{K}$  can be decomposed as

$$(3.38) \quad \mathcal{K} \leftrightarrow \mathfrak{h}(\mathcal{K}) \otimes \Gamma_{inv}^{\wedge} \quad \mathfrak{h}(\mathcal{K}) = \mathcal{K} \cap \mathfrak{h}(\Omega).$$

Let us consider the spectral sequence  $E_{\mathcal{K}}$  associated to  $\mathcal{K}$ . From the form of the second cohomology algebra  $E_{\mathcal{K}}^2 = H(\mathcal{K} \cap \mathfrak{I}) \otimes H(G)$  we conclude that

**Proposition 3.17.** *The first non-trivial components of  $H(\mathcal{K})$  and  $H(\mathcal{K} \cap \mathfrak{I})$  coincide.  $\square$*

As we have seen,  $H^{2k-1}(\mathfrak{J}) = \{0\}$  for each  $k \in \mathbb{N}$ . On the other hand  $\mathfrak{I}$  will generally contain non-trivial odd-dimensional classes. From the exact sequence (3.37) it follows that

$$(3.39) \quad H^{2k-1}(\mathcal{K} \cap \mathfrak{I}) \twoheadrightarrow H^{2k-1}(\mathfrak{I}) \quad H^{2k}(\mathcal{K} \cap \mathfrak{I}) \hookrightarrow H^{2k}(\mathfrak{I}).$$

A particularly interesting special case is when the universal characteristic classes for the regular and the general case coincide.

**Proposition 3.18.** *The following conditions are equivalent:*

- (i) *The complex  $\mathcal{K} \cap \mathfrak{I}$  is acyclic.*
- (ii) *The complex  $\mathcal{K}$  is acyclic.*
- (iii) *We have  $H(\Omega_*) = \mathbb{C}$ .*
- (iv) *The projection map  $p_{\mathcal{K}}: \mathfrak{I} \rightarrow \mathfrak{J}$  induces the isomorphism of cohomology algebras  $H(\mathfrak{I})$  and  $H(\mathfrak{J})$ .*

*Proof.* The previous lemma implies that (i) and (ii) are equivalent. Considering the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \Omega \rightarrow \Omega_* \rightarrow 0,$$

and remembering that  $H^k(\Omega) = \{0\}$  for  $k \in \mathbb{N}$  we conclude that  $H^k(\Omega_*) = H^{k+1}(\mathcal{K})$  and hence the equivalence between (ii) and (iii) holds. Finally, looking at the exact sequence (3.37) we conclude that (iv) and (i) are equivalent.  $\square$

The isomorphism between cohomology algebras of  $\mathbb{T}$  and  $\mathbb{Q}$  allows the essential technical simplification in constructing quantum characteristic classes—it is sufficient to perform algebraic constructions assuming the regular case, dealing with the braided symmetric polynomials.

The spectral sequence  $E(\Omega)$  converges to the trivial cohomology  $H(\Omega) = \mathbb{C}$ . Therefore the spaces  $H^k(\mathbb{T})$  are naturally filtered. The filtration is induced from the sequences of the form

$$0 \rightarrow [\ ]_j^{-1} \left\{ \text{im}(d_{j-1}) \right\} \rightarrow H(\mathbb{T}) \rightarrow E_j^{*0}(\Omega) \rightarrow 0$$

The trivial convergence information is insufficient to compute the cohomology of  $H(\mathbb{T})$ . However, in various interesting special cases the spectral sequence degenerates (as in classical geometry), and the triviality property is sufficient to determine all cohomology classes.

#### 4. QUANTUM CHERN CHARACTER

Through this section we shall deal with structures admitting regular and multiplicative connections. The aim is to construct the analogue of the Chern character. This requires introducing the appropriate  $K$ -rings. We shall start from the algebra of horizontal forms, and extract vector bundles as intertwining bimodules (playing the role of differential forms on  $M$  with values in associated vector bundles). These bimodules generate a ring  $K(M, P)$ . The Chern character will be constructed with the help of the covariant derivative (of an arbitrary regular and multiplicative connection), following classical analogy and imposing a formula similar to that of [1]. In particular, we shall introduce a canonical “trace” in the algebras of corresponding left module homomorphisms. In such a way we obtain the Chern character as a right module structure  $\text{ch}_M: HZ(M) \times K(M, P) \rightarrow HZ(M)$ .

It is important to point out a difference between this  $\text{ch}_M$  and the Chern character introduced in [1]. The later is defined on the standard algebraic  $K$ -group (generated by finite projective right  $\mathcal{V}$ -modules), and has values in the cyclic homology of  $M$ . In particular, it is inherently associated to the space  $M$ , because it depends exclusively on the algebra  $\mathcal{V}$ . On the other hand the ring  $K(M, P)$  explicitly depends on the bundle  $P$ , and also on differential structures on  $P$  and  $G$  (the later influences the calculus on  $M$ ). Furthermore, in contrast to the classical case,  $\text{ch}_M$  is not interpretable as a homomorphism between  $K(M, P)$  and  $HZ(M)$ , because generally elements from these two spaces do not commute (as operators of right/left multiplication acting in  $HZ(M)$ ). This leads to a variety of interesting purely quantum phenomenas. In the last subsection we sketch a construction of quantum Chern classes.

#### 4.1. The Canonical Trace

For each  $u \in R(G)$  let  $\text{tr}_M: \text{ghom}(\mathcal{F}_u) \rightarrow \Omega(M)$  be a linear map defined via the following diagram

$$(4.1) \quad \begin{array}{ccc} \mathcal{E}_u^* \otimes_M \mathcal{E}_u & \xrightarrow{\text{id} \otimes A} & \mathcal{E}_u^* \otimes_M \mathcal{E}_u \otimes_M \Omega(M) \\ \gamma_\star^u \uparrow & & \downarrow \langle \rangle_u^- \\ \mathcal{V} \hookrightarrow \mathcal{E}_\emptyset & \xrightarrow{\text{tr}_M(A)} & \mathcal{V} \otimes_M \Omega(M) \hookrightarrow \Omega(M) \end{array}$$

In the above diagram,  $\text{tr}_M(A)$  is a left  $\mathcal{V}$ -module homomorphism, and hence it can be equivalently understood as an element of  $\Omega(M)$ . If  $A: \mathcal{E}_u \rightarrow \mathcal{F}_u$  is a  $\mathcal{V}$ -bimodule homomorphism then

$$f \text{tr}_M(A) = \text{tr}_M(A) f$$

for each  $f \in \mathcal{V}$ . Moreover,

$$\text{tr}_M[\text{ghom}(\mathcal{F}_u)] \subseteq Z(M).$$

It is worth noticing that

$$\text{tr}_M(f_\star) = \text{tr}(C_u^{-1} f)$$

for each  $f \in \text{Mor}(u, u)$ .

The following equalities hold

$$(4.2) \quad \text{tr}_M(A \oplus B) = \text{tr}_M(A) + \text{tr}_M(B)$$

$$(4.3) \quad \text{tr}_M(wA) = \text{tr}_M(A)w,$$

where  $A, B \in \text{ghom}(\mathcal{F}_{u,v})$  and  $w \in \Omega(M)$  is understood as the operator of right multiplication. If moreover  $A \in \text{ghom}(\mathcal{E}_u, \mathcal{F}_u)$  then

$$(4.4) \quad \text{tr}_M(A \otimes B) = \text{tr}_M(\text{tr}_M(A)B).$$

The above equalities directly follow from the definition of  $\text{tr}_M$ .

**Lemma 4.1.** *The pairings  $\langle \rangle_u^+: \mathcal{E}_u \otimes_M \mathcal{E}_u^* \rightarrow \mathcal{V}$  and  $\langle \rangle_u^-: \mathcal{E}_u^* \otimes_M \mathcal{E}_u \rightarrow \mathcal{V}$  induce, in a natural manner, bimodule isomorphisms of the form*

$$(4.5) \quad \text{ghom}(\mathcal{E}_u, \mathcal{V}) \hookrightarrow \mathcal{E}_u^* \quad \text{rhom}(\mathcal{E}_u, \mathcal{V}) \hookrightarrow \mathcal{E}_u^*.$$

*Proof.* We shall check the first isomorphism, the second one follows by a similar reasoning. As first, the map  $\langle \rangle_u^+$  naturally induces a bimodule homomorphism  $\chi_u^+: \mathcal{E}_u^* \rightarrow \text{ghom}(\mathcal{E}_u, \mathcal{V})$ . This map is injective, because of the identity

$$\sum_k \nu_k \langle \mu_k, \psi \rangle_u^+ = \psi$$

where  $\sum_k \nu_k \otimes \mu_k = \gamma_*^u(1)$ . If  $\varphi: \mathcal{E}_u \rightarrow \mathcal{V}$  is an arbitrary left  $\mathcal{V}$ -linear map then the element  $\psi = \sum_k \nu_k \varphi(\mu_k)$  satisfies

$$\begin{aligned} \chi_u^+(\psi)(\xi) &= \langle \xi, \psi \rangle_u^+ = \sum_k \langle \xi, \nu_k \varphi(\mu_k) \rangle_u^+ = \sum_k \langle \xi, \nu_k \rangle_u^+ \varphi(\mu_k) \\ &= \sum_k \varphi \left[ \langle \xi, \nu_k \rangle_u^+ \mu_k \right] = \varphi(\xi), \end{aligned}$$

for each  $\xi \in \mathcal{E}_u$ . In other words,  $\chi_u^+(\psi) = \varphi$ , and hence  $\chi_u^+$  is bijective.  $\square$

The symmetry between  $u$  and  $\bar{u}$  implies that also

$$(4.6) \quad \text{lhom}(\mathcal{E}_u^*, \mathcal{V}) \leftrightarrow \mathcal{E}_u \quad \text{rhom}(\mathcal{E}_u^*, \mathcal{V}) \leftrightarrow \mathcal{E}_u,$$

where the isomorphisms are induced by the same contraction maps. More generally, for each  $u, v \in R(G)$  the following bimodule isomorphisms hold

$$(4.7) \quad \mathcal{E}_u^* \otimes_M \mathcal{E}_v \leftrightarrow \text{lhom}(\mathcal{E}_u, \mathcal{E}_v) \quad \mathcal{E}_u \otimes_M \mathcal{E}_v^* \leftrightarrow \text{rhom}(\mathcal{E}_v, \mathcal{E}_u).$$

Furthermore, the identification

$$(u \times v)^c = v^c \times u^c \quad (H_u \otimes H_v)^* = H_v^* \otimes H_u^*$$

induces the following natural bimodule isomorphism

$$(4.8) \quad \mathcal{E}_{u \times v}^* \leftrightarrow \mathcal{E}_v^* \otimes_M \mathcal{E}_u^*,$$

so that the following identities hold

$$\begin{aligned} \langle x \otimes y, \psi \otimes \varphi \rangle_{u \times v}^+ &= \langle x \langle y, \psi \rangle_v^+ \varphi \rangle_u^+ \\ \langle \psi \otimes \varphi, x \otimes y \rangle_{u \times v}^- &= \langle \psi \langle \varphi, x \rangle_u^- y \rangle_v^-. \end{aligned}$$

The same analysis is applicable to  $\Omega(M)$ -bimodules  $\mathcal{F}_u$ . If  $A \in \text{lhom}(\mathcal{F}_u, \mathcal{F}_v)$  then its *right transposed* operator  $A^\top \in \text{rhom}(\mathcal{F}_v^*, \mathcal{F}_u^*)$  is defined in a natural manner. Similarly, each  $A \in \text{rhom}(\mathcal{F}_u, \mathcal{F}_v)$  induces a map  $A_\perp \in \text{lhom}(\mathcal{F}_v^*, \mathcal{F}_u^*)$ . Explicitly,

$$(4.9) \quad \langle \varphi, Ax \rangle_v^- = \langle A_\perp \varphi, x \rangle_u^- \quad \text{for } A \in \text{rhom}(\mathcal{F}_u, \mathcal{F}_v)$$

$$(4.10) \quad \langle Ax, \varphi \rangle_v^+ = \langle x, A^\top \varphi \rangle_u^+ \quad \text{for } A \in \text{lhom}(\mathcal{F}_u, \mathcal{F}_v).$$

Maps  $\top: \text{lhom}(\mathcal{F}_u, \mathcal{F}_v) \rightarrow \text{rhom}(\mathcal{F}_v^*, \mathcal{F}_u^*)$  and  $\perp: \text{rhom}(\mathcal{F}_u, \mathcal{F}_v) \rightarrow \text{lhom}(\mathcal{F}_v^*, \mathcal{F}_u^*)$  are bimodule anti-isomorphisms. Moreover,  $\top$  and  $\perp$  act as mutually inverse transformations and

$$(4.11) \quad (AB)_\perp = B_\perp A_\perp \quad (AB)^\top = B^\top A^\top,$$

if  $A, B$  are appropriate right/left  $\Omega(M)$ -module homomorphisms.

In particular, if  $A \in \text{hom}(\mathcal{E}_u, \mathcal{F}_v)$  then both transposed maps are defined.

**Definition 4.1.** An element  $A \in \text{hom}(\mathcal{E}_u, \mathcal{F}_v)$  is called *transposable* iff  $A_\perp = A^\top$ .

If  $A$  is transposable then  $A^\top \in \text{hom}(\mathcal{E}_v^*, \mathcal{F}_u^*)$  is transposable too. From (4.11) it follows that the composition of transposable homomorphisms is again transposable.

It is important to mention that if  $f \in \text{Mor}(u, v)$  then  $f_\star: \mathcal{F}_v \rightarrow \mathcal{F}_u$  is transposable. In this case

$$(4.12) \quad (f_\star)^\top = (f^\top)_\star.$$

Finally, the tensor product of transposable bimodule homomorphisms is transposable. This property follows from the identities

$$(4.13) \quad (A \otimes B)^\top = B^\top \otimes A^\top \quad (A \otimes B)_\perp = B_\perp \otimes A_\perp$$

where the identification (4.8) is assumed.

**Lemma 4.2.** *We have*

$$(4.14) \quad \text{tr}_M(AB) = (-1)^{\partial A \partial B} \text{tr}_M(BA),$$

for every  $A \in \text{h}(\mathcal{F}_u)$  and transposable  $B \in \text{gh}(\mathcal{F}_u)$ .

*Proof.* Let us suppose that  $A \leftrightarrow \varphi \otimes x$ , with  $\varphi \in \mathcal{E}_u^*$  and  $x \in \mathcal{F}_u$ . In terms of this identification we have  $BA - (-1)^{\partial A \partial B} AB \leftrightarrow \varphi \otimes B(x) - B^\top(\varphi) \otimes x$  and hence

$$\text{tr}_M(BA - (-1)^{\partial A \partial B} AB) = \langle \varphi, B(x) \rangle_u^- - \langle B^\top(\varphi), x \rangle_u^- = 0. \quad \square$$

Now, we are going to study analogs of covariant derivative maps which play an important role in the construction of the Chern character. We shall denote by  $\text{lder}(\mathcal{F}_u)$  the space of first-order linear maps  $D: \mathcal{F}_u \rightarrow \mathcal{F}_u$  satisfying

$$(4.15) \quad D(\alpha\psi) = d(\alpha)\psi + (-1)^{\partial\alpha} \alpha D(\psi)$$

for each  $\psi \in \mathcal{F}_u$  and  $\alpha \in \Omega(M)$ . Similarly,  $\text{rder}(\mathcal{F}_u)$  will refer to first-order maps  $D: \mathcal{F}_u \rightarrow \mathcal{F}_u$  with the property

$$(4.16) \quad D(\psi\alpha) = D(\psi)\alpha + (-1)^{\partial\psi} \psi d(\alpha).$$

Finally, we define

$$(4.17) \quad \text{der}(\mathcal{F}_u) = \text{lder}(\mathcal{F}_u) \cap \text{rder}(\mathcal{F}_u).$$

If  $D \in \text{rder}(\mathcal{F}_u)$  then the formula

$$(4.18) \quad d\langle \varphi, \psi \rangle_u^- = \langle D_\perp(\varphi), \psi \rangle_u^- + (-1)^{\partial\varphi} \langle \varphi, D(\psi) \rangle_u^-$$

consistently and uniquely defines a map  $D_\perp \in \text{lder}(\mathcal{F}_u^*)$ . Similarly, for each  $D \in \text{lder}(\mathcal{F}_u)$  the formula

$$(4.19) \quad d\langle \psi, \varphi \rangle_u^+ = \langle D(\psi), \varphi \rangle_u^+ + (-1)^{\partial\psi} \langle \psi, D^\top(\varphi) \rangle_u^+$$

defines a transposed map  $D^\top \in \text{rder}(\mathcal{F}_u^*)$ . The two transposed operations are understandable as mutually inverse, in a natural manner. In particular, for  $D \in \text{der}(\mathcal{F}_u)$  both transposed maps are defined.

**Definition 4.2.** We say that  $D \in \text{der}(\mathcal{F}_u)$  is *transposable* iff  $D^\top = D_\perp$ .

If  $D$  is transposable then  $D^\top \in \text{der}(\mathcal{F}_u^*)$  is transposable too. Furthermore, if  $D_1 \in \text{der}(\mathcal{F}_u)$  and  $D_2 \in \text{der}(\mathcal{F}_v)$  are transposable then  $D: \mathcal{F}_{u \times v} \rightarrow \mathcal{F}_{u \times v}$  given by

$$D(\vartheta \otimes \eta) = D_1(\vartheta) \otimes \eta + (-1)^{\partial \vartheta} \vartheta \otimes D_2(\eta)$$

is also transposable and (modulo the identification (4.8)) we have

$$D^\top(\varphi \otimes \psi) = D_2^\top(\varphi) \otimes \psi + (-1)^{\partial \varphi} \varphi \otimes D_1^\top(\psi).$$

If  $\omega$  is a regular connection on  $P$  then every map  $D_{\omega, u}: \mathcal{F}_u \rightarrow \mathcal{F}_u$  is transposable, with

$$D_{\omega, u}^\top = D_{\omega, \bar{u}}.$$

Let us consider an arbitrary  $D \in \text{lder}(\mathcal{F}_u)$ . The formula

$$(4.20) \quad \nabla(A) = DA - (-1)^{\partial A} AD$$

defines a graded derivation  $\nabla$  on  $\text{lhom}(\mathcal{F}_u)$ . If in addition  $D \in \text{der}(\mathcal{F}_u)$  then the algebras  $\text{rhom}(\mathcal{F}_u)$  and  $\text{ghom}(\mathcal{F}_u)$  are  $\nabla$ -invariant.

**Lemma 4.3.** *If  $D \in \text{der}(\mathcal{F}_u)$  is transposable then*

$$(4.21) \quad d_M \text{tr}_M(A) = \text{tr}_M \nabla(A)$$

for each  $A \in \text{lhom}(\mathcal{F}_u)$ .

*Proof.* It is sufficient to check the above formula for elements of the form  $A \leftrightarrow \varphi \otimes x$ . In this case  $DA - (-1)^{\partial A} AD \leftrightarrow D^\top(\varphi) \otimes x + (-1)^{\partial \varphi} \varphi \otimes D(x)$  and we have

$$\begin{aligned} d_M \text{tr}_M(A) &= d_M \langle \varphi, x \rangle_u^- = \langle D^\top \varphi, x \rangle_u^- + (-1)^{\partial \varphi} \langle \varphi, Dx \rangle_u^- \\ &= \text{tr}_M(D^\top(\varphi) \otimes x + (-1)^{\partial \varphi} \varphi \otimes D(x)) = \text{tr}_M(DA - (-1)^{\partial A} AD) = \text{tr}_M \nabla(A). \quad \square \end{aligned}$$

For a fixed  $u \in R(G)$ , let us consider expressions

$$(4.22) \quad \Theta_n(A, D) = \text{tr}_M(AD^{2n}),$$

where  $A \in Z(M)$  and  $D \in \text{der}(\mathcal{F}_u)$ . It is worth noticing that the map  $D^2$  is a bimodule homomorphism. This implies that  $\Theta_n(A, D) \in Z(M)$ .

**Lemma 4.4.** (i) *Let us assume that  $D$  is transposable. We have*

$$(4.23) \quad d[\Theta_n(A, D)] = \Theta_n(d(A), D),$$

and hence it is possible to pass to the cohomology classes  $HZ(M)$  in the first argument.

(ii) *At the level of cohomology classes,  $\Theta_n(A, D)$  does not depend of the choice of  $D$ .*



*Proof.* The first statement directly follows from Lemma 4.3. Let us consider another transposable  $D' \in \text{der}(\mathcal{F}_u)$ , and let

$$D_t = D + tS$$

be a 1-parameter family of transposable elements, where  $S = D' - D$  is a transposable element of  $\text{ghom}(\mathcal{F}_u)$ . Let us assume that  $A$  is closed. We have then

$$\begin{aligned} \frac{d}{dt}\Theta_n(A, D_t) &= \frac{d}{dt}\text{tr}_M(AD_t^{2n}) = \sum_{i=1}^{2n} \text{tr}_M(AD_t^{i-1}SD_t^{2n-i}) \\ &= \text{tr}_M(D_tX - (-1)^{\partial X}XD_t) = d[\text{tr}_M(X)], \end{aligned}$$

where

$$X = \sum_{i=1}^n D_t^{2i-2}SD_t^{2n-2i}A.$$

It follows that  $\Theta_n(A, D)$  and  $\Theta_n(A, D')$  belong to the same cohomology class.  $\square$

Let us now fix a regular connection  $\omega$  on  $P$ , and consider covariant derivatives  $D_{\omega, u}: \mathcal{F}_u \rightarrow \mathcal{F}_u$ . Let  $\text{ch}^u: Z(M) \rightarrow Z(M)$  be a linear map given by

$$(4.24) \quad \text{ch}^u = \sum_{k=0}^{\infty} \frac{1}{(2\pi i)^k k!} \Theta_k(A, D_{\omega, u}).$$

Strictly speaking, the above infinite sum should be interpreted as a formal series in a graded algebra.

**Proposition 4.5.** (i) *The following identities hold*

$$(4.25) \quad \text{ch}^{u \oplus v}(A) = \text{ch}^u(A) + \text{ch}^v(A)$$

$$(4.26) \quad \text{ch}^{u \times v}(A \otimes B) = \text{ch}^v(\text{ch}^u(A)B).$$

(ii) *We have*

$$(4.27) \quad d[\text{ch}^u(A)] = \text{ch}^u(d(A)).$$

*The induced cohomology maps  $\text{ch}^u: HZ(M) \rightarrow HZ(M)$  are independent of the choice of  $\omega$ .*

*Proof.* Property (4.25) directly follows from (4.2). Similarly (4.26) follows from (4.4) and the Leibniz rule for  $D^2$ . We have

$$\begin{aligned} \text{ch}^{u \times v}(A \otimes B) &= \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^{2n} n!} \text{tr}_M((A \otimes B)D_{\omega, u \times v}^{2n}) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{(2\pi i)^{2n} n!} \binom{n}{k} \text{tr}_M(AD_{\omega, u}^{2k} \otimes BD_{\omega, v}^{2n-2k}) \\ &= \sum_{kl=0}^{\infty} \frac{1}{(2\pi i)^{k+l}} \frac{1}{k!l!} \text{tr}_M(\text{tr}_M(AD_{\omega, u}^{2k})BD_{\omega, v}^{2l}) = \text{ch}^v(\text{ch}^u(A)B). \end{aligned}$$

The statement (ii) follows from Lemma 4.4.  $\square$

## 4.2. Associated $K$ -rings & Chern Characters

Let  $K(G)$  be the  $K$ -ring associated to the semiring of equivalence classes of  $R(G)$  (with the induced operations of addition and multiplication). As we have seen, each representation gives rise to the cohomology map  $\mathbf{ch}^u: HZ(M) \rightarrow HZ(M)$ . Equivalent representations give the same map. Formulas (4.25)–(4.26) imply that maps  $\mathbf{ch}^u$  induce a global map of the form

$$(4.28) \quad \mathbf{ch}: HZ(M) \times K(G) \rightarrow HZ(M) \quad ([w], [u]) \mapsto [\mathbf{ch}^u(w)],$$

which is a right  $K(G)$ -module structure on  $HZ(M)$ .

The ring  $K(G)$  generally contains a lot of redundancy, from the point of view of the introduced module structure on  $HZ(M)$ . We shall now introduce a relativized analogue of the  $K$ -ring for  $M$ .

Let  $\Phi(M, P)$  be the parametrized set of bimodules  $\mathcal{F}_u$ , where  $u \in R(G)$ . Let  $\sim$  be an equivalence relation on  $\Phi(M, P)$  defined in the following way.

We say that  $\mathcal{F}_u \sim \mathcal{F}_v$  iff there exists a transposable isomorphism  $J_{uv}: \mathcal{F}_u \rightarrow \mathcal{F}_v$ . By definition, the introduced equivalence relation is compatible with the product and the sum of bimodules. Let  $V(M, P) = \Phi(M, P)/\sim$  be the semiring of corresponding equivalence classes.

**Lemma 4.6.** *If  $\mathcal{F}_u \sim \mathcal{F}_v$  then*

$$(4.29) \quad \mathrm{tr}_M(A) = \mathrm{tr}_M(J_{uv}AJ_{uv}^{-1})$$

for each  $A \in \mathrm{Lhom}(\mathcal{F}_u)$ .

*Proof.* Modulo the natural identifications  $\mathcal{F}_u^* \otimes_M \mathcal{F}_u = \mathrm{Lhom}(\mathcal{F}_u)$  we have

$$(4.30) \quad \mathrm{tr}_M(A) = \langle A \rangle_u^-.$$

It is sufficient to check the formula (4.29) for operators of the form  $A = \varphi \otimes x$ . We have

$$J_{uv}AJ_{uv}^{-1}(y) = \langle J_{uv}^{-1}(y), \varphi \rangle_u^+ J_{uv}(x) = \langle y, (J_{uv}^{-1})^\top \varphi \rangle_v^+ J_{uv}(x)$$

which implies

$$\mathrm{tr}_M(J_{uv}AJ_{uv}^{-1}) = \langle (J_{uv}^{-1})^\top \varphi, J_{uv}(x) \rangle_v^- = \langle \varphi, x \rangle_u^- = \mathrm{tr}_M(A). \quad \square$$

Let  $K(M, P)$  be the  $K$ -ring associated to  $V(M, P)$ . By construction there exists a natural ring epimorphism  $p_M: K(G) \rightarrow K(M, P)$ , given by

$$p_M([u]) = [\mathcal{F}_u].$$

The construction of the map  $\mathbf{ch}: HZ(M) \times K(G) \rightarrow HZ(M)$  can be factorized through  $p_M$ , in other words

**Lemma 4.7.** *There exists the unique map  $\mathbf{ch}_M: HZ(M) \times K(M, P) \rightarrow HZ(M)$  such that the diagram*

$$(4.31) \quad \begin{array}{ccc} HZ(M) \times K(G) & \xrightarrow{\mathbf{ch}} & HZ(M) \\ \mathrm{id} \times p_M \downarrow & & \downarrow \mathrm{id} \\ HZ(M) \times K(M, P) & \xrightarrow{\mathbf{ch}_M} & HZ(M) \end{array}$$

is commutative.

*Proof.* If  $D \in \text{der}(\mathcal{F}_u)$  is transposable then  $J_{uv}DJ_{uv}^{-1}$  is also transposable, as follows from transposability of  $J_{uv}$ . The statement now follows from Lemma 4.6. In other words,  $HZ(M)$  is a right  $K(M, P)$ -module, in a natural manner.  $\square$

**Definition 4.3.** The map  $\text{ch}_M$  is called *the Chern character* for  $M$ , relative to  $P$ .

### 4.3. Explicit Expressions

We are going to find an explicit formula for the Chern character, in terms of the curvature map. To simplify the notation, we shall use the product symbol  $\circ$  for the corresponding  $K(M, P)$ -module structure on  $HZ(M)$ . For each  $A \in \text{hom}(\mathcal{F}_u)$  let  $A_u$  be a matrix given by

$$(4.32) \quad (A_u)_{ij} = \sum_{lk} (C_u^{-1})_{il} \nu_k(e_l^*) (A\mu_k)(e_j).$$

We have then

$$(4.33) \quad \text{tr}_M(A) = \text{tr}(A_u)$$

where on the right-hand side of the above equality figures the standard trace, and  $\sum_k \nu_k \otimes \mu_k \leftrightarrow I \in \text{hom}(\mathcal{F}_u)$ . For an arbitrary regular connection  $\omega$ , let  $R_{\omega, u}$  be a  $\mathfrak{hot}(P)$ -valued matrix defined by

$$(R_{\omega, u})_{ij} = R_\omega \pi(u_{ij}).$$

**Lemma 4.8.** (i) *The following identity holds*

$$(4.34) \quad [A] \circ [\mathcal{F}_u] = \sum_{\alpha} \frac{(-1)^{\alpha}}{(2\pi i)^{\alpha} \alpha!} [\text{tr}(A_u R_{\omega, u}^{\alpha})].$$

(ii) *The algebra  $\mathfrak{w}(M)$  is invariant under the action of  $K(M, P)$ .*

*Proof.* Property (i) follows from the expression

$$D^2 \mu(e_i) = - \sum_j \mu(e_j) (R_{\omega, u})_{ji}$$

and equality (4.33). To prove (ii), we shall find an explicit formula for the action of  $K(M, P)$  on elements of  $\mathfrak{w}(M)$ . If  $\vartheta \in I^*$  then

$$\begin{aligned} [R^{\otimes}(\vartheta)] \circ [\mathcal{F}_u] &= \sum_{\alpha} \frac{(-1)^{\alpha}}{(2\pi i)^{\alpha} \alpha!} \left[ \sum_{ilkn} (C_u^{-1})_{li} \nu_k(e_i^*) R^{\otimes}(\vartheta) \mu_k(e_n) (R_{\omega, u}^{\alpha})_{nl} \right] \\ &= \sum_{\alpha} \frac{(-1)^{\alpha}}{(2\pi i)^{\alpha} \alpha!} \left[ \sum_{ilkjn} (C_u^{-1})_{li} \nu_k(e_i^*) \mu_k(e_j) R^{\otimes}(\vartheta \circ u_{jn}) (R_{\omega, u}^{\alpha})_{nl} \right] \\ &= \sum_{\alpha} \frac{(-1)^{\alpha}}{(2\pi i)^{\alpha} \alpha!} \left[ \sum_{jln} (C_u^{-1})_{lj} R^{\otimes}(\vartheta \circ u_{jn}) (R_{\omega, u}^{\alpha})_{nl} \right], \end{aligned}$$

which evidently belongs to  $\mathfrak{w}(M)$ .  $\square$

#### 4.4. Chern Classes

In this subsection we present a general construction of quantum Chern classes. As in the whole section, we work in the regular picture, however the constructed universal Chern classes are straightforwardly incorporable into the context of arbitrary bundles, as far as  $H(\Omega_*) = \mathbb{C}$ . This follows from Proposition 3.18. However, for bundles admitting regular connections, Chern classes can be viewed as invariants of associated vector bundles, as in classical geometry.

Let  $\mathcal{A}^{\text{ad}} \subseteq \mathcal{A}$  be the subalgebra of ad-invariant elements. Let  $\Delta: \mathcal{A}^{\text{ad}} \rightarrow I(\Sigma)_\lambda$  be a map given by

$$(4.35) \quad \Delta^\lambda(a) = \exp \left\{ \sum_{k \in \mathbb{N}} \lambda^k \frac{\pi(a^{(1)}) \dots \pi(a^{(k)})}{k} \right\},$$

where  $I(\Sigma)_\lambda$  is the algebra of  $\lambda$ -series with coefficients in  $I(\Sigma)$ . We shall assume that  $\lambda$  is real.

**Lemma 4.9.** *The following identities hold*

$$(4.36) \quad \Delta^\lambda(a + b) = \Delta^\lambda(a) \Delta^\lambda(b)$$

$$(4.37) \quad \Delta^\lambda[\kappa(a)^*] = [\Delta^{-\lambda}(a)]^*.$$

*Proof.* To prove the first equality, it is sufficient to recall that  $I(\Sigma)$  is commutative. The second equality follows by a direct computation:

$$\begin{aligned} \Delta^\lambda[\kappa(a)^*] &= \exp \left\{ \sum_{k \in \mathbb{N}} \lambda^k \frac{\pi[\kappa(a^{(k)})^*] \dots \pi[\kappa(a^{(1)})^*]}{k} \right\} \\ &= \exp \left\{ \sum_{k \in \mathbb{N}} (-\lambda)^k \frac{\pi(a^{(k)})^* \dots \pi(a^{(1)})^*}{k} \right\} = [\Delta^{-\lambda}(a)]^*. \quad \square \end{aligned}$$

The coefficients in the expansion

$$(4.38) \quad \Delta^\lambda(a) = \sum_{n \geq 0} \lambda^n c_n(a),$$

play the role of actual universal Chern classes. According to the previous lemma

$$(4.39) \quad c_n(a + b) = \sum_{k+l=n} c_k(a) c_l(b)$$

$$(4.40) \quad c_n(a)^* = (-1)^n c_n[\kappa(a)^*].$$

For each  $u \in R(G)$  let us denote by  $\chi_u \in \mathcal{A}^{\text{ad}}$  its *character*. Explicitly,

$$(4.41) \quad \chi_u = \text{tr}(C_u^{-1}u).$$

As in the classical theory, we have

$$\chi_{u \oplus v} = \chi_u + \chi_v \quad \chi_{u \times v} = \chi_u \chi_v,$$

and the characters of irreducible representations form a basis in the space  $\mathcal{A}^{\text{ad}}$ .

Let us assume that  $\omega$  is a regular connection on a quantum principal bundle  $P$ . Then the maps  $c_n$  can be further factorized through the equivalence relation determining  $V(M, P)$ . Indeed, we have

$$(4.42) \quad W\Delta^\lambda(\chi_u) = \left[ \exp \operatorname{tr}_M \left\{ \sum_{k \in \mathbb{N}} (-\lambda)^k D_{\omega, u}^k \right\} \right],$$

and hence the formula

$$c_n[\mathcal{F}_u] = c_n(\chi_u)$$

consistently defines maps  $c_n : V(M, P) \rightarrow HZ(M)$ , playing the role of Chern classes of associated vector bundles.

It is interesting to observe that  $\Delta^\lambda(a)$  is generally infinite as a  $\lambda$ -series, in contrast to the classical case.

## 5. QUANTUM LINE BUNDLES

In this section a structural analysis of quantum line bundles will be performed, including the study of the internal structure of general differential calculi on them corresponding to a 1-dimensional calculus on the structure group  $G = U(1)$ . Finally, we present a construction of the quantum Euler class for arbitrary quantum principal bundles admitting regular connections.

If  $G = U(1)$  then the algebra  $\mathcal{A}$  is generated by a single unitary element  $u \in \mathcal{A}$ , satisfying  $\phi(u) = u \otimes u$ . Irreducible representations of  $G$  are labeled by integers  $n \leftrightarrow u^n$ .

### 5.1. Reconstruction Problematics

The structural analysis of general quantum principal bundles presented in [7] essentially simplifies in the case of quantum line bundles. Let  $P = (\mathcal{B}, i, F)$  be such a bundle over a quantum space  $M$ .

The decomposition to multiple irreducible submodules is simply

$$(5.1) \quad \mathcal{B} = \sum_{n \in \mathbb{Z}}^{\oplus} \mathcal{B}_n, \quad \mathcal{B}_n = \left\{ b \in \mathcal{B} \mid F(b) = b \otimes u^n \right\}$$

so that we have

$$(5.2) \quad (\mathcal{B}_n)^* = \mathcal{B}_{-n} \quad \mathcal{B}_{n+m} = \mathcal{B}_n \otimes_M \mathcal{B}_m,$$

in particular the product map in  $\mathcal{B}$  induces  $\mathcal{V}$ -bimodule isomorphisms of the form

$$\mu_+ : \mathcal{E} \otimes_M \mathcal{E}^* \rightarrow \mathcal{V} \quad \mu_- : \mathcal{E}^* \otimes_M \mathcal{E} \rightarrow \mathcal{V},$$

where  $\mathcal{E} = \mathcal{B}_1$  and  $\mathcal{E}^* = \mathcal{B}_{-1}$  correspond to the line bundle (and its conjugate) canonically associated to  $P$ . Both maps are hermitian, in a natural manner. The structure of the algebra  $\mathcal{B}$  is encoded in  $\{\mathcal{V}, \mathcal{E}, \mathcal{E}^*, \mu_{\pm}\}$ .

In terms of identification (5.1) the product in  $\mathcal{B}$  is the standard tensor multiplication, factorized through the identifications induced by maps  $\mu_{\pm}$ . The  $*$ -structure

on  $\mathcal{B}$  is uniquely determined by the  $*$ -structure on  $\mathcal{V}$ , and the conjugation between  $\mathcal{E}$  and  $\mathcal{E}^*$ . The associativity of the product in  $\mathcal{B}$  implies that

$$(5.3) \quad \begin{aligned} \text{id} \otimes \mu_+ &= \mu_- \otimes \text{id}: \mathcal{E}^* \otimes_M \mathcal{E} \otimes_M \mathcal{E}^* \rightarrow \mathcal{V} \\ \text{id} \otimes \mu_- &= \mu_+ \otimes \text{id}: \mathcal{E} \otimes_M \mathcal{E}^* \otimes_M \mathcal{E} \rightarrow \mathcal{V}. \end{aligned}$$

Conversely, the above properties are sufficient to construct the bundle  $P$ , starting from  $\mathcal{E}$  and maps  $\mu_{\pm}$ . Let us assume that a  $\mathcal{V}$ -bimodule  $\mathcal{E}$  is given, and let  $\mathcal{E}^*$  be the conjugate bimodule. Let us assume that there exist hermitian  $\mathcal{V}$ -bimodule isomorphisms  $\mu_-: \mathcal{E}^* \otimes_M \mathcal{E} \rightarrow \mathcal{V}$  and  $\mu_+: \mathcal{E} \otimes_M \mathcal{E}^* \rightarrow \mathcal{V}$  such that properties (5.3) hold.

Let us define first bimodules  $\mathcal{B}_n$ , and then the algebra  $\mathcal{B}$  by (5.1), at the level of vector spaces. Let the product in  $\mathcal{B}$  be induced by the tensor product and identifications given by maps  $\mu_{\pm}$ . Properties (5.3) ensure the associativity of the introduced product. By construction, the  $*$ -structure on  $\mathcal{V}$  and the mutual conjugation between  $\mathcal{E}$  and  $\mathcal{E}^*$  admit the unique extension to a  $*$ -structure on  $\mathcal{B}$ .

Let  $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$  be a  $*$ -homomorphism given by  $F(q) = q \otimes u^n$ , where  $q \in \mathcal{B}_n$ . By construction  $\mathcal{V}$  is the fixed-point subalgebra for this action. Let  $i: \mathcal{V} \hookrightarrow \mathcal{B}$  be the canonical inclusion map.

**Lemma 5.1.** *The triplet  $P = (\mathcal{B}, i, F)$  is a principal  $G$ -bundle over  $M$ .  $\square$*

## 5.2. Differential Structures and The Euler Class

Let us assume that the calculus on  $G$  is 1-dimensional and non-standard, and that higher-order components of  $\Gamma^{\wedge}$  are trivial. Calculi  $\Gamma$  satisfying the above assumptions are given by ideals of the form

$$\mathcal{R} = \left(u^{-1} + \frac{u}{\lambda} - \left(1 + \frac{1}{\lambda}\right)\right)\mathcal{A},$$

where  $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$ . The reality of  $\lambda$  ensures the existence of the  $*$ -structure on  $\Gamma$ . The element  $\zeta = \pi(u)/(1 - \lambda)$  spans the space  $\Gamma_{inv}^{\wedge}$ , and we have

$$\zeta \circ a = p_{\lambda}(a)\zeta$$

where  $p_{\lambda}: \mathcal{A} \rightarrow \mathbb{C}$  is the character specified by  $p_{\lambda}(u) = \lambda$ . The ideal  $S_{inv}^{\wedge}$  is generated by  $\zeta \otimes \zeta$ , and consequently  $\Gamma_{inv}^{\wedge} = \mathbb{C} \oplus \Gamma_{inv}$ .

Let  $P$  be an arbitrary quantum principal  $G$ -bundle over  $M$  and let  $\Omega(P)$  be a graded-differential  $*$ -algebra describing the full calculus on  $P$ . Let us fix a connection  $\omega$  on  $P$ .

According to the general theory each element  $w \in \Omega(P)$  can be uniquely decomposed as

$$w = \varphi + \psi\omega$$

where  $\varphi, \psi \in \mathfrak{hor}(P)$  with  $\varphi = h_{\omega}(w)$ , and we have identified  $\omega = \omega(\zeta)$ . This enables us to write

$$(5.4) \quad \Omega(P) = \mathfrak{hor}(P) \oplus \mathfrak{hor}(P)$$

at the level of left  $\mathfrak{hor}(P)$ -modules.

We pass to the study of the representation of the differential  $*$ -algebra structure on  $\Omega(P)$ , in terms of the above identification. Let  $\ell: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$  be a linear map defined by

$$(5.5) \quad \ell(\varphi) = \omega\varphi - (-1)^{\partial\varphi}\chi(\varphi)\omega,$$

where  $\chi: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$  is an automorphism given by

$$(5.6) \quad \chi(\varphi) = (\text{id} \otimes p_\lambda)F^\wedge(\varphi).$$

The first-order map  $\ell = \ell_\omega(\zeta, *)$  measures the lack of regularity of  $\omega$ . It satisfies a  $\chi$ -relativized graded Leibniz rule

$$(5.7) \quad \ell(\varphi\psi) = \ell(\varphi)\psi + (-1)^{\partial\varphi}\chi(\varphi)\ell(\psi).$$

The following covariance properties hold

$$(5.8) \quad \begin{aligned} \chi(\varphi)^* &= \chi^{-1}(\varphi^*) & F^\wedge\chi &= (\chi \otimes \text{id})F^\wedge \\ \ell(\varphi)^* &= \ell\chi^{-1}(\varphi^*) & F^\wedge\ell &= (\ell \otimes \text{id})F^\wedge. \end{aligned}$$

In terms of the realization (5.4), the product in  $\Omega(P)$  is given by

$$(5.9) \quad \begin{aligned} (w, u)(\varphi, \psi) &= \\ &= (w\varphi + u\ell(\varphi) + (-1)^{\partial\psi}u\chi(\psi)\rho, u\ell(\psi) + (-1)^{\partial\varphi}u\chi(\varphi) + w\psi + (-1)^{\partial\psi}u\chi(\psi)). \end{aligned}$$

Here  $\rho = \omega^2 \in \Omega(M)$  measures the lack of multiplicativity of  $\omega$ . The  $*$ -structure on  $\Omega(P)$  is specified by

$$(5.10) \quad (w, u)^* = (w^* - (-1)^{\partial u}\ell(u^*), -\chi(u^*)).$$

Let  $D: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$  be the covariant derivative associated to  $\omega$ , and  $R = d\omega$  the corresponding curvature. From the general theory it follows that the following identities hold

$$(5.11) \quad D(\varphi\psi) = D(\varphi)\psi + (-1)^{\partial\varphi}\varphi D(\psi) + (-1)^{\partial\varphi}(\varphi - \chi(\varphi))\ell(\psi)$$

$$(5.12) \quad -D^2(\varphi) = (I - \chi)(\varphi)R + (I - \chi)^2(\varphi)\rho.$$

Furthermore, the differential map is given by

$$(5.13) \quad \begin{aligned} d(w, u) &= \\ &= (D(w) + (-1)^{\partial u}uR + (-1)^{\partial u}[u - \chi(u)]\rho, D(u) + (-1)^{\partial w}[w - \chi(w)]). \end{aligned}$$

The pull back map  $\widehat{F}$  is representable as follows

$$(5.14) \quad \widehat{F}(w, u) = (F^\wedge(w) + F^\wedge(u)\zeta, F^\wedge(u)),$$

where  $F^\wedge: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P) \otimes \mathcal{A}$  is the right action map.

Finally, the following additional algebraic properties hold:

$$(5.15) \quad \begin{aligned} R^* &= -R & \rho^* &= -\rho & D(\varphi)^* &= D(\varphi^*) + \ell(\varphi^*) - \ell(\varphi)^* \\ \ell(\rho) &= 0 & d\rho &= -\ell(R) & \rho\varphi &= \ell^2(\varphi) + \chi^2(\varphi)\rho \\ D\ell(\varphi) + \ell D(\varphi) &= R\varphi - \chi(\varphi)R - 2\chi(\varphi - \chi(\varphi))\rho & DR &= 0. \end{aligned}$$

This completes the structural analysis of  $\Omega(P)$ . We see that the calculus is completely determined by maps  $\{D, *, F^\wedge, \chi, \ell\}$  acting in  $\mathfrak{hor}(P)$ , and second-order differential forms  $\{R, \rho\}$  on the base manifold.

Conversely, the derived algebraic properties can be taken as a starting point in constructing the calculus on  $P$ . Let us assume that a graded  $*$ -algebra  $\mathfrak{hor}_P$  representing horizontal forms is given, such that  $\mathfrak{hor}_P^0 = \mathcal{B}$ . Let us further assume that  $\mathfrak{hor}_P$  is endowed with maps  $D, \chi, \ell: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$  and the extension  $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$  of the right action  $F$ . Let  $\Omega_M \subseteq \mathfrak{hor}_P$  be the fixed-point subalgebra for the action  $F^\wedge$ , and let  $R, \rho$  be second-order elements of  $\Omega_M$ . Finally, let us assume that all the above derived algebraic properties hold. Then formulas (5.9), (5.10) and (5.13) determine a differential  $*$ -algebra structure on  $\Omega_P = \mathfrak{hor}_P \oplus \mathfrak{hor}_P$ . Moreover, the formula

$$(5.16) \quad \omega(\zeta) = (0, 1)$$

determines a connection  $\omega$ . The “reconstruction circle” is closed by observing that  $\{R, \rho, D, \ell\}$  can be re-expressed in terms of  $\omega$ , in the original way.

Let us analyze the cohomology algebra  $H(\mathbb{T})$  for the given 1-dimensional  $\Gamma$ . By definition,  $\Omega$  is the free differential algebra generated by a first order generator  $\zeta$ . The map  $\tilde{\omega}$  acts as follows

$$(5.17) \quad \tilde{\omega}(\zeta) = \zeta \otimes 1 + 1 \otimes \zeta \quad \tilde{\omega}(d\zeta) = d\zeta \otimes 1.$$

The space of  $d$ -cocycles of  $\Omega^+$  is spanned by the elements  $w = d(\zeta^{n_1}) \dots d(\zeta^{n_k})$ .

**Lemma 5.2.** *We have*

$$(5.18) \quad H^{2k}(\mathbb{T}) = \mathbb{C} \quad H^{2k+1}(\mathbb{T}) = \{0\}$$

for each  $k \in \mathbb{N} \cup \{0\}$ . More precisely, the algebra  $H(\mathbb{T})$  is generated by the second-order element  $d\zeta$ .

*Proof.* It follows by analyzing the structure of the differential algebra  $E_2(\Omega)$ , taking into account that  $H(G) = \mathbb{C} \oplus \mathbb{C}$ , and observing that the spectral sequence stabilizes at the third term.  $\square$

The algebra of characteristic classes is therefore generated by the cohomological class  $e_M = [d\omega]$  in  $\Omega(M)$ , which is a direct counterpart of the Euler class. The connection form  $\omega$  is here a counterpart of the global angular form [3].

Let us assume that the bundle admits regular connections  $\omega$ , and find the action of  $K(M, P)$  on the elements of a subalgebra  $\mathfrak{w}(M) \subseteq HZ(M)$ . We have

$$\pi(u^n) = (1 - \lambda^n)\zeta$$

for each  $n \in \mathbb{Z}$ , and a straightforward application of the explicit formulas of Subsection 4.3 gives

$$(5.19) \quad (e_M^k) \circ [\mathcal{F}_n] = \lambda^{nk} \sum_{\alpha \geq 0} \frac{(\lambda^n - 1)^\alpha}{(2\pi i)^\alpha \alpha!} e_M^{k+\alpha}$$

where  $n$  labels non-trivial irreducible representations.



To conclude this section, we shall present a general construction of quantum Euler class, at the level of bundles  $P$  possessing regular connections. We follow [11], the structure group  $G$  is arbitrary.

Let  $u \in R(G)$  be a representation satisfying  $u \sim u^c$ . Let  $S: H_u \rightarrow H_u^*$  be an equivalence between  $u$  and  $u^c$ . The formula

$$\lambda_{u,S}(e_i \otimes e_j) = \sum_k S_{ki} u_{kj} \quad S(e_i) = \sum_k S_{ki} e_k^*$$

defines a map  $\lambda_{u,S}: H_u \otimes H_u \rightarrow \mathcal{A}$  intertwining  $u \times u$  and the adjoint action  $\text{ad}$ .

Let us now assume that there exists a braid operator  $\tau: H_u^{\otimes 2} \rightarrow H_u^{\otimes 2}$  which intertwines  $u \times u$ , and such that there exists a *volume element* in the corresponding braided exterior algebra  $H_u^\wedge$ . In other words, there exists  $k \in \mathbb{N}$  such that  $H_u^{\wedge k}$  is 1-dimensional, while higher order components vanish. Let us consider a volume element  $w \in H_u^{\wedge k} \setminus \{0\}$ . We have

$$u^\wedge(w) = w \otimes \Delta_u$$

where  $\Delta_u \in \mathcal{A}$  is the corresponding quantum determinant and  $u^\wedge$  the corresponding representation of  $G$  in  $H_u^\wedge$ . It is worth noticing that  $k$  is not necessarily equal to the dimension of  $H_u$ .

Let us finally assume that  $k = 2n$  and that  $u$  is *unimodular*, in the sense that  $\Delta_u = 1$ . Let  $w_\Sigma \in I(\Sigma)$  be an element given by

$$(5.20) \quad w_\Sigma = c_n [\pi^{\otimes n} \lambda_{u,S}^{\otimes n}(w)]_\Sigma, \quad c_n > 0.$$

**Definition 5.1.** The characteristic class

$$(5.21) \quad e_M^u = W(w_\Sigma)$$

is called the *quantum Euler class* of  $P$ , relative to  $u$ .

## 6. BUNDLES OVER CLASSICAL MANIFOLDS

In this section we shall study characteristic classes of locally trivial bundles over classical smooth manifolds, applying the theory developed in [5]. Let  $M$  be a compact smooth manifold,  $P$  a locally trivial principal  $G$ -bundle over  $M$ , and let us assume that  $\Gamma$  is the minimal admissible (bicovariant  $*$ -) calculus over  $G$ . The admissibility here means that  $\Gamma$  is compatible, in appropriate sense, with all local retrivializations (transition functions) of the bundle. Let  $\Omega(P)$  be a graded-differential  $*$ -algebra canonically associated [5] to  $P$  and  $\Gamma$ . The main property of  $\Omega(P)$  is its full *local trivializability*, in the sense that all local trivializations of the bundle locally trivialize the calculus, too.

Let  $\tau = (\pi_U)_{U \in \mathcal{U}}$  be an arbitrary trivialization system for  $P$ . Locally, connection forms are (uniquely) expressed via the corresponding “gauge fields” as

$$\pi_U^\wedge \omega(\vartheta) = (A_U \otimes \text{id}) \varpi(\vartheta) + 1_U \otimes \vartheta,$$

where  $A_U: \Gamma_{inv} \rightarrow \Omega(U)$  are hermitian maps, mutually related in the appropriate way on overlappings of regions from  $\mathcal{U}$ . Here  $\pi_U^\wedge: \Omega(P) \rightarrow \Omega(U) \hat{\otimes} \Gamma^\wedge$  are the corresponding trivializations of the calculus. The curvature is expressed by

$$(6.1) \quad \pi_U^\wedge R_\omega(\vartheta) = (F_U \otimes \text{id})\varpi(\vartheta), \quad F_U = dA_U - \langle A_U, A_U \rangle.$$

Regular connections are characterized by

$$(6.2) \quad A_U(\vartheta \circ a) = \epsilon(a)A_U(\vartheta),$$

for each  $\vartheta \in \Gamma_{inv}$  and  $a \in \mathcal{A}$ .

According to [5] quantum  $G$ -bundles  $P$  are in a 1–1 correspondence with classical  $G_{cl}$ -bundles (over the same manifold), where  $G_{cl}$  is the *classical part* of  $G$  (consisting of classical points of  $G$ ). Explicitly, the elements of  $P_{cl}$  are in a natural correspondence with classical points of  $P$  (the  $*$ -characters of  $\mathcal{B}$ ).

Regularity condition (6.2) is equivalent to the statement that  $A_U$  are factorizable through the canonical projection map  $\nu: \Gamma_{inv} \rightarrow \text{lie}(G_{cl})^*$ . In this sense regular connections are understandable as connections on  $P_{cl}$  (classical connections, in the terminology of [5]). Moreover, the curvature operator “commutes” with the restriction from  $P$  to  $P_{cl}$ . This follows from equality

$$F_U(\vartheta \circ a) = \epsilon(a)F_U(\vartheta).$$

In summary, characteristic classes of  $P$  are interpretable as (classical) characteristic classes of  $P_{cl}$ . The converse is generally not true, because  $\nu$  generally maps invariant elements of  $\Sigma$  into a proper subalgebra of invariant polynomials over  $\text{lie}(G_{cl})$ . The classical nature of quantum characteristic classes is explained by the correspondence  $P \leftrightarrow P_{cl}$ . The whole topological nontriviality of  $P$  is already contained in its classical part  $P_{cl}$ .

The above described concept of local triviality can be further naturally generalized in two ways [4, 12]. One way of generalizing consists in replacing  $M$  with an appropriate quantum space, endowed with a system of noncommutative “open sets”  $U$ , locally trivializing  $P$ . Secondly, it is possible to relax the requirement that locally the bundle and the calculus are tensor products of algebras—it is sufficient to ask that locally  $\Omega(P) \leftrightarrow \Omega(U) \otimes \Gamma^\wedge$ , but with the appropriate crossed product structure (so that locally  $\hat{F} \leftrightarrow \text{id} \otimes \hat{\phi}$ ). Local expressions for  $\omega$  and  $R_\omega$  will be essentially the same in this general context.

Furthermore, requiring that all local trivializations of the bundle locally trivialize the calculus implies an “admissibility” constraint on the calculus  $\Gamma$ , and there exists the minimal calculus (for a given bundle  $P$ ) resolving the constraints at the first-order level. In both choices  $\{\Gamma^\wedge, \Gamma^\vee\}$  for the higher-order calculus on  $G$ , compatibility with retrivializations will be automatical, at the higher-order levels.

However, even when the base is classical, the class of locally trivial bundles is too restrictive. As an interesting example, we shall describe a locally-nontrivializable quantum line bundle over a classical compact manifold  $M$ .

Let us consider an arbitrary quantum line bundle  $P$  over  $M$ , and assume that  $\mathcal{V} = S(M)$ . At the level of left modules, we can write  $\mathcal{E} = S(\xi)$  in a natural manner, where  $\xi$  is a line vector bundle (in the standard sense) over  $M$ . In terms of this identification, the right module structure is described by

$$\psi f = \varepsilon(f)\psi$$

where  $\varepsilon: \mathcal{V} \rightarrow \mathcal{V}$  is a  $*$ -automorphism. This map further extends to the automorphism  $\varepsilon: \Omega(M) \rightarrow \Omega(M)$ , where  $\Omega(M)$  is the standard graded-differential  $*$ -algebra of smooth differential forms. Actually,  $\varepsilon$  is induced (via the pull back) by a diffeomorphism  $\varepsilon_*: M \rightarrow M$ .

The algebra  $\mathcal{B}$  is noncommutative, if  $\varepsilon \neq I$ . If the line bundle  $\xi$  is non-trivial, and if the map  $\varepsilon_*$  is such that  $\{\emptyset, M\}$  are the only  $\varepsilon_*$ -invariant open subsets of  $M$  then the bundle  $P$  is not locally trivial.

This example is also interesting from the point of view of characteristic classes. Let  $\mathfrak{hor}(P)$  be the  $*$ -algebra of horizontal forms, built from  $\{\xi, \varepsilon, \Omega(M)\}$ .

**Proposition 6.1.** *There exists a natural correspondence between regular covariant derivative maps  $D: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$  and standard connections  $\nabla$  on  $\xi$ .  $\square$*

Let  $\Phi \in \Omega^2(M)$  be the curvature of  $\nabla$ . Let  $\Gamma$  be the minimal calculus on  $G = U(1)$  compatible with  $D$ , in the sense of [8]. Let  $\Omega(P)$  be the corresponding graded-differential  $*$ -algebra representing the calculus on  $P$ , constructed applying the construction of [6]-Section 6. Finally, let  $\omega: \Gamma_{inv} \rightarrow \Omega(P)$  be the canonical regular connection on  $P$ , so that we have  $D = D_\omega$ . The curvature of this connection is given by

$$\begin{aligned} R_\omega \pi(u^n) &= \varepsilon^{-1}(\Phi) + \cdots + \varepsilon^{-n}(\Phi) \\ -R_\omega \pi(u^{-n}) &= \Phi + \cdots + \varepsilon^{n-1}(\Phi), \end{aligned}$$

where  $n \in \mathbb{N}$ .

In the general case, the resulting calculus  $\Gamma$  will be multi-dimensional. The algebra  $\mathfrak{w}(M)$  of characteristic classes will be generated by the above expressions. In other words, it is the minimal  $\varepsilon$ -invariant subalgebra of  $H(M)$  containing the class of  $\Phi$ .

## 7. FRAMED BUNDLES

In general, there exist no natural prescriptions of constructing differential calculi over a given quantum principal bundle  $P$ . However, when  $P$  is equipped with the appropriate additional structure, it will be possible to associate a distinguished differential algebra  $\Omega(P)$  to the bundle. A large class of structures of this kind giving non-trivial calculi is given by framed bundles [9, 10]. Let us sketch the construction of a natural calculus on such bundles.

Let  $\Psi$  be a bicovariant  $*$ -bimodule over  $G$ . According to the general theory [14], it is structuralized as  $\Psi \leftrightarrow \mathcal{A} \otimes V$ , where  $V = \Psi_{inv}$ .

Let  $v: V \rightarrow V \otimes \mathcal{A}$  and  $\circ$  be the natural right action and the right  $\mathcal{A}$ -module structure on  $V$ , respectively. Let  $\tau$  be the canonical braid operator on  $V \otimes V$ . In what follows it will be assumed that  $\ker(I + \tau) \neq \{0\}$ .

Let  $V^\wedge$  be the corresponding  $\tau$ -exterior algebra, obtained by factorizing  $V^\otimes$  through the relations  $\text{im}(I + \tau)$ . The  $*$ -structure,  $v$  and  $\circ$  are naturally extendible to  $V^\wedge$ , so that we have a bicovariant graded  $*$ -algebra  $\Psi^\wedge \leftrightarrow \mathcal{A} \otimes V^\wedge$ .

Let us consider a graded vector space

$$\mathfrak{hor}_P = \mathcal{B} \otimes V^\wedge$$

endowed with the following  $*$ -algebra structure

$$\begin{aligned} (q \otimes \vartheta)(b \otimes \eta) &= \sum_i q b_i \otimes (\vartheta \circ c_i) \eta \\ (b \otimes \vartheta)^* &= \sum_i b_i^* \otimes (\vartheta^* \circ c_i^*), \end{aligned}$$

where  $\sum_i b_i \otimes c_i = F(b)$ .

There exists the unique action  $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$  extending actions  $F$  and  $v^\wedge: V^\wedge \rightarrow V^\wedge \otimes \mathcal{A}$ . Let us observe that the above definition of the product in  $\mathfrak{hor}_P$ , together with the definition of  $V^\wedge$ , implies the following commutation relations

$$\vartheta \varphi = (-1)^{\partial \varphi \partial \vartheta} \varphi_k (\vartheta \circ c_k),$$

with  $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$  and  $\vartheta \in V^\wedge$ .

By definition, the bundle  $P$  is  $\Psi$ -framed, iff there exists a first-order antiderivation  $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$  satisfying

$$\begin{aligned} F^\wedge D &= (D \otimes \text{id}) F^\wedge & *D &= D* \\ D(V^\wedge) &= \{0\} & D^2(\mathcal{V}) &= \{0\}, \end{aligned}$$

and such that the canonical map  $\sharp_D: \mathcal{B} \otimes \mathcal{V} \rightarrow \mathfrak{hor}^1(P)$  given by  $\sharp_D(b \otimes f) = bD(f)$  is surjective.

Let  $\Omega_M \subseteq \mathfrak{hor}_P$  be the  $F^\wedge$ -fixed point subalgebra. The covariance condition implies  $D(\Omega_M) \subseteq \Omega_M$ . Let  $d_M: \Omega_M \rightarrow \Omega_M$  be the restriction map. By construction,  $\Omega_M^0 = \mathcal{V}$ . Furthermore, it follows that  $d_M^2 = 0$ . Moreover it turns out that  $\Omega_M$  is generated by  $\mathcal{V}$ , as a differential algebra equipped with  $d_M$ .

Applying to  $\{\mathfrak{hor}_P, F^\wedge, D, \Omega_M\}$  a general construction presented in [8] we obtain in the intrinsic way a graded-differential  $*$ -algebra  $\Omega(P)$  representing the calculus on  $P$ , a bicovariant  $*$ -calculus  $\Gamma$  over  $G$ , and a regular and multiplicative connection  $\omega: \Gamma_{inv} \rightarrow \Omega(P)$  such that  $\mathfrak{hor}_P$  is recovered as the corresponding  $*$ -subalgebra of horizontal forms for  $\Omega(P)$ , and such that  $D = D_\omega$ . The connection  $\omega$  corresponds to the Levi-Civita connection in classical geometry.

## 8. CONCLUDING EXAMPLES & REMARKS

### *Odd-dimensional Classes*

As we have mentioned in Section 3, the algebra  $\mathbb{T}$  may have non-trivial odd-dimensional cohomology classes. We shall here describe an interesting example giving  $H^3(\mathbb{T}) \neq \{0\}$ . As first, we are going to prove that the higher-order calculus on  $G$  can be always maximally adopted, to allow the appearance of 3-dimensional characteristic classes.

As we have seen, 2-dimensional Chern-Simons forms are labeled by closed  $\{\varpi, \sigma\}$ -invariant elements of  $\Gamma_{inv}^{*2}$ . Let  $S_3 \subseteq \Gamma^\wedge$  be the ideal generated by the elements of the form  $d^\wedge(\psi)$ , where  $\psi \in \Gamma_{inv}^{\wedge 2}$  satisfy  $\sigma(\psi) = \psi$  and  $\varpi(\psi) = \psi \otimes 1$ . By definition,  $S_3$  is a graded-differential  $*$ -ideal.

**Lemma 8.1.** *The ideal  $S_3$  is  $\widehat{\phi}$ -invariant, in other words*

$$(8.1) \quad \widehat{\phi}(S_3) \subseteq S_3 \widehat{\otimes} \Gamma^\wedge + \Gamma^\wedge \widehat{\otimes} S_3.$$

Therefore we can pass to the factor-calculus  $\Gamma^\Delta = \Gamma^\wedge / S_3$ , in the framework of which all  $\{\varpi, \sigma\}$ -invariant elements of  $\Gamma_{inv}^{\wedge 2}$  are closed.

As a concrete illustration, let us assume that  $G$  is a compact simple simply connected Lie group. The elements of  $G$  are naturally interpretable as characters of  $\mathcal{A}$ . Let us denote by  $\mathfrak{g}$  the corresponding Lie algebra. Let us consider a calculus  $\Gamma$  over  $G$  defined in the following way. We put

$$(8.2) \quad \Gamma_{inv} = [\mathfrak{g}^*]^{Z(G)},$$

with  $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$  given by taking standard differentials in points of  $Z(G)$ . In other words  $\mathcal{R}$  is consisting of elements of  $\ker(\epsilon)$  the derivatives of which vanish on  $Z(G)$ .

By construction,  $\Gamma$  is a bicovariant  $*$ -calculus. The adjoint action  $\varpi$  is induced by the classical  $\varpi$  on  $\mathfrak{g}$ . The  $*$ -structure is a combination of the classical  $*$  and the inverse  $g \leftrightarrow g^{-1}$ , acting on the coordinate set  $Z(G)$ . There exists a natural left action of  $Z(G)$  on  $\Gamma_{inv}$ , given by left translations on  $Z(G)$ . Furthermore, the  $\circ$ -structure is given by

$$(8.3) \quad [\psi \circ a]_g = g(a)[\psi]_g,$$

where  $g \in Z(G)$ . It follows that the braiding  $\sigma$  coincides with the standard transposition, in particular  $\Gamma^\vee$  is the standard exterior algebra. However  $\Gamma^\wedge \neq \Gamma^\vee$ . Let us compute the space  $S_{inv}^{\wedge 2}$ . Acting by  $(\pi \otimes \pi)\phi$  on the elements of  $\mathcal{R}$  we conclude that

**Lemma 8.2.** *The universal envelope is given by the relations*

$$(8.4) \quad S_{inv}^{\wedge 2} = \ker(I - \sigma) \cap \left\{ \psi \in \Gamma_{inv}^{\otimes 2} \mid (g \otimes g^{-1})\psi = \psi \quad \forall g \in Z(G) \right\}.$$

In particular, there exist  $\varpi$ -singlets in the space  $\ker(I - \sigma)/S_{inv}^{\wedge 2}$  of  $\sigma$ -invariant elements of  $\Gamma_{inv}^{\wedge 2}$ . Finally, let us assume that the higher-order calculus on  $G$  is described by  $\Gamma^\Delta$ . Having in mind that  $\varpi: \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathcal{A}$  is irreducible it follows that

$$(8.5) \quad H^3(\mathbb{T}) = \left\{ \{\varpi, \sigma\}\text{-invariant elements of } \Gamma_{inv}^{\wedge 2} \right\}.$$

Actually in this case all  $\varpi$ -invariant elements will be  $\sigma$ -invariant.

#### *Some Particular Cases*

Let us consider the trivial principal bundle  $P$  over a 1-point set  $M = \{*\}$ , with the trivial differential calculus. Then  $\mathcal{F}_u = H_u^*$ , in a natural manner. The corresponding contraction maps are given by

$$\langle \varphi, x \rangle_u^+ = \varphi(x) \quad \langle x, \varphi \rangle_u^- = \varphi C_u^{-1}(x).$$

Let us consider a linear map  $A: H_u^* \rightarrow H_v^*$ . Its right transposed coincides with the standard transposed  $A^\top$ , while

$$A_\perp = C_u A^\top C_v^{-1}.$$

Hence,  $A$  will be transposable iff  $A^\top$  intertwines  $C_u$  and  $C_v$ . In particular,  $\mathcal{F}_u \sim \mathcal{F}_v$  iff  $C_u$  and  $C_v$  have the same eigenvalues with the same multiplicities.

The formalism includes non-trivial examples with differential calculus over such trivial spaces. However, it may be necessary to refine the above introduced equivalence relation on  $R(G)$ , taking the presence of  $\Omega(M)$  into account.

For example, we can choose  $\Omega(P) = \Gamma^\wedge$  and assume that the calculus on  $G$  is described by  $\Gamma^\vee$ . In this case we have

$$\widehat{F} = (\text{id} \otimes \mathbb{J})\widehat{\phi}.$$

Actually, the algebra  $\Omega(M)$  measures in a certain sense the difference between algebras  $\Gamma^\wedge$  and  $\Gamma^\vee$ . It follows that

$$\Omega(M) = \mathbb{C} \quad \Leftrightarrow \quad \Gamma^\wedge = \Gamma^\vee.$$

If  $P$  is a line bundle and if the calculus on  $G$  is 1-dimensional then the spectral sequence always stabilizes at the third term. Hence, it is possible to pass to a long exact sequence of cohomology groups

$$\rightarrow H^n(P) \rightarrow H^{n-1}(M) \rightarrow H^{n+1}(M) \rightarrow H^{n+1}(P) \rightarrow$$

as in classical geometry. In particular, if  $H(P) = \mathbb{C}$  then  $H(M)$  will be freely generated by the Euler class.

#### *Non-Standard Chern Classes*

A simple example of the appearance of infinitely many universal Chern classes is given by  $G = S_\mu U(2)$ , with a  $4D$ -calculus  $\Gamma$ . Let us consider this in more details. We have

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \quad C_u = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}$$

and the space  $\Gamma_{inv}$  is spanned by elements

$$\tau = \pi(\mu^2\alpha + \alpha^*) \quad \eta_3 = \pi(\alpha - \alpha^*) \quad \eta_- = \pi(\gamma^*) \quad \eta_+ = \pi(\gamma),$$

forming  $\varpi$ -singlet and triplet respectively. Using explicit formulas [5] for the braiding  $\sigma$  it follows that  $\Sigma$  is generated by the relations

$$\tau\eta_i = \eta_i\tau = -\frac{(1-\mu^3)\varkappa_i}{(1-\mu^2)(1+\mu)},$$

where

$$\begin{aligned} \varkappa_+ &= \eta_+ \otimes \eta_3 - \mu^2\eta_3 \otimes \eta_+ & \varkappa_- &= \eta_3 \otimes \eta_- - \mu^2\eta_- \otimes \eta_3 \\ \varkappa_3 &= (1-\mu^2)\eta_3 \otimes \eta_3 + \mu(1+\mu^2)(\eta_+ \otimes \eta_- - \eta_- \otimes \eta_+). \end{aligned}$$

Let  $\Sigma_3 \subseteq \Sigma$  be the subalgebra generated by triplet elements  $\eta_i$ . The above formulas imply that  $\Sigma_3$  is generated by the following cubic relations

$$\eta_i\varkappa_j = \varkappa_i\eta_j, \quad i, j \in \{+, 3, -\}.$$

Every element  $w \in \Sigma$  can be uniquely decomposed in the form

$$w = p(\tau) + q(\eta_+, \eta_3, \eta_-),$$

where  $p, q$  are polynomial expressions. The simplest way to prove the non-triviality of Chern classes in high orders is to factorize  $\Sigma$  through the ideal generated by  $\{\eta_+, \eta_-\}$ . The factor-algebra has two generators  $\tau, \eta_3$ , satisfying a single relation

$$\eta_3 \tau = \tau \eta_3 = -\frac{1 - \mu^3}{1 + \mu} \eta_3^2.$$

A direct calculation gives

$$\begin{aligned} \Delta^\lambda(\chi_u) &= \\ &= \left(1 + \frac{\tau\lambda}{1 + \mu^2}\right)^{\mu+1/\mu} + \left(1 + \frac{\mu\lambda\eta_3}{1 + \mu}\right)^\mu \left(1 - \frac{\eta_3\lambda}{1 + \mu}\right)^{1/\mu} - \left(1 - \frac{(1 - \mu^3)\lambda\eta_3}{(1 + \mu)(1 + \mu^2)}\right)^{\mu+1/\mu} \end{aligned}$$

for the factorized  $\Delta$ .

#### *Equivalence of Vector Bundles*

As we have already mentioned, the introduced equivalence relation  $\sim$  between vector bundles  $\mathcal{F}_u$  depends also on the specification of the calculus (via the algebra  $\mathfrak{h}\mathfrak{o}\mathfrak{r}(P)$ ). The additional structural element coming from the calculus is expressed via canonical flip-over operators  $\sigma_u: \mathcal{E}_u \otimes_M \Omega(M) \leftrightarrow \Omega(M) \otimes_M \mathcal{E}_u$ . Homomorphisms acting on pure sections  $\mathcal{E}_u$  will be extendible to the higher-order level iff they commute with these flip-over operators.

Covariant derivative maps  $D \in \text{rhom}(\mathcal{F}_u)$  are in a natural correspondence with their restrictions  $D: \mathcal{E}_u \rightarrow \mathcal{E}_u \otimes_M \Omega^1(M)$ , characterized as first-order linear maps satisfying

$$D(\psi f) = D(\psi)f + \psi \otimes f,$$

for each  $\psi \in \mathcal{E}_u$  and  $f \in \mathcal{V}$ . In other words, the maps  $D$  are *connections* on a finite projective module  $\mathcal{E}_u$ , in the sense of [1]. A given connection  $D$  will satisfy the left Leibniz rule iff the following compatibility condition holds

$$[\sigma_u D \sigma_u^{-1}](\alpha \otimes \psi) = d\alpha \otimes \psi + (-1)^{\partial\alpha} \alpha \sigma_u D(\psi),$$

where  $\alpha \in \Omega(M)$  and  $\psi \in \mathcal{E}_u$ .

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